ANALYSIS OF NONLINEAR DYNAMIC BEHAVIOUR OF NANOBEAM RESTING ON WINKLER AND PASTERNAK FOUNDATIONS USING VARIATIONAL ITERATION METHOD

M. G. Sobamowo[1]* and G. A. Oguntala[2]

[1] Department of Mechanical Engineering, Faculty of Engineering, University of Lagos, Akoka, Lagos, Nigeria.

[2] School of Electrical Engineering and Computer Science, Faculty of Engineering and Informatics, University of Bradford, UK

ABSTRACT

Dynamic modeling of nanobeam under stretching and two-parameter foundations effects result in nonlinear equations that are very difficult to find exact analytical solutions. In this study, variational iteration method is used to develop approximate analytical solutions to nonlinear vibration analysis of nanobeam under the effects of stretching, Winkler and Pasternak foundations. The governing equation of motion for the nanobeam was derived based on Euler-Bernoulli beam theory. The developed approximate analytical solutions for the governing equation are used to study the effects of the model parameters on the dynamic behaviour of the nanobeam. The results show that increase in the beam length decreases the natural frequency of vibration while the diameter of the nanobeam increases as the natural frequency increases. As the spring constant increases, the nonlinear frequency ratio decreases. At a high stiffness media, the carbon nanobeam behavior can be modeled as a linear system whose geometric nonlinearity becomes negligible. The nonlinear frequency of nanobeam increases with increase in the vibration amplitude and the discrepancy between the linear and nonlinear responses tends to increase as time evolves. Also, it is found that as the foundation parameter increases, the nonlinear vibration frequency ratio increases and the difference between the nonlinear and linear frequency becomes pronounced. These analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as they provide clearer insights to understanding the problems in comparison with numerical methods.

Keyword: *Nonlinear vibration; Variational Iteration method; Nanobeam; Winkler and Pasternak foundations.*

*Corresponding author e-mail: mikegbeminiyiprof@yahoo.com

1.0 INTRODUCTION

The study of dynamic behaviour of nanobeam is an important research area due to small scale of carbon nanobeam (CNB) and their important applications in sensitive devices. The vibration behaviour and properties of the CNB have been investigated in the past few decades. Many of the past researches are based on linear vibration analysis of the CNB. However, owing to the small scale of carbon nanobeams, the linear assumptions cannot provide an accurate prediction and analysis of the vibration of the CNB. Furthermore, the assumption that carbon nanobeam rests on linear foundations shows apparently that the linear foundation is not very precise approximation for the tiny instruments, and so the obtained past estimations either by numerical or analytical approaches cannot accurately predict the dynamic behaviour of the CNB. Therefore, in order to increase the level of prediction or accurately predict the dynamic behaviour of CNB, it is very essential to develop mathematical model for the CNB, which contains geometrical nonlinearity and nonlinear foundations. Modeling the nanobeam under such considerations results in nonlinear dynamic equations which are difficult to solve exactly and analytically. However, in many cases under different scenarios, recourse is always made to numerical methods to solve the nonlinear or approximate analytical methods are often applied in which their accuracies largely depend on the number of terms included in the solutions. In some cases, where decomposition procedures into spatial and temporal parts are carried out, the resulting nonlinear equation for the temporal part comes in form of Duffing equation. Application of exact analytical methods to the nonlinear equation is limited as many of the cases where the exact solutions are generated are not practicable and the solutions hardly provide an all-encompassing understanding of the nature of systems in response to parameters affecting nonlinearity. However, the classical way for finding analytical solution either exact or approximated is obviously still very important since it serves as an accurate benchmark for numerical solutions. Although, different approximate analytical methods such as Perturbation method (regular or singular perturbation method), homotopy perturbation method (HPM), Homotopy analysis method (HAM), variational iterative method (VIM), differential transformation method (DTM), Harmonic balancing method, Adomian decomposition method etc. Zhou (1986); Liao, (1992), (1995); Chen and Ho. (1996), He, (1998), Momani, (2004), El-Shahed, (2008); Liao and Tan, (2007), Fernandez (2009); Rafiepour et al. (2014). These approximate analytical methods solve nonlinear differential equations without linearization, without discretization or approximation of the derivatives. However, most of the approximate methods give accurate predictions only when the nonlinearities are weak and they fail to predict accurate solutions for strong nonlinear models. Also, when they are routinely implemented, they can sometimes lead to erroneous results (Sobamowo, 2016). Additionally, some of them require more mathematical manipulations and are not applicable to all problems, and thus suffer a lack of generality. For example, DTM proved to be more effective than most of the other approximate analytical solutions as it does not require many computations as carried out in ADM, HAM, HPM, and VIM. However, the transformation of the nonlinear equations and the development of equivalent recurrence equations for the nonlinear equations using DTM proved somehow difficult in some nonlinear system such as in rational Duffing oscillator, irrational nonlinear Duffing oscillator, finite extensibility nonlinear oscillator. Therefore, the quest for comparatively simple, flexible, generic and high accurate analytical solutions continues. Moreover, the determination of Adomian polynomials as carried out in ADM, the restrictions of HPM to weakly nonlinear problems, the lack of rigorous theories or proper guidance for choosing initial approximation, auxiliary linear operators, auxiliary functions, and auxiliary parameters in HAM, operational restrictions to small domains and the search for a particular value for the auxiliary parameter that will satisfy second the boundary condition which leads to additional computational cost in using DTM, HAM, ADM. In the class of the approximate analytical methods, the relative simplicity and flexibility of VIM makes it a desirable and promising method for the analysis of nonlinear problems. The method has been applied to solve many nonlinear problems (He, 1998a, 1998b, 1999a, 1999b, 2000, 2006, 2007a, 2007b, 2011, 2012a, 2012b; 2012c; Rafei et al. ,2007, Marinca and Herisanu, 2006; Ganji, et al., 2008; Hesameddini and Latifizadeh, 2009; Wu, 2012). Therefore, in this work, variation iteration method (VIM) is applied to develop approximate analytical solutions for nonlinear vibration analysis of single-walled carbon nanobeam under the effects of stretching and Winkler and Pasternak foundations. Variational iteration method has shown to be the one of the most effective, accurate, flexible, convenient approximate analytical methods for large class of weakly and strongly nonlinear equations. It is a user-friendly method with reduced size of calculation, direct and straightforward iteration and generates solution with a rapid rate of convergent and without any restrictive assumptions or transformations. In VIM, the initial solution can be freely chosen with some unknown parameters and the unknown parameters in the initial solution can be achieved easily. Although, there is a rigour of step-by-step integrations coupled with the problem of determination of Lagrange multiplier in application of VIM, with few number of iteration, even, in some cases, a single iteration of VIM can converge to correct solutions or results**.** The analytical solutions as developed in this work can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problem than pure numerical or computation methods.

2.0 PROBLEM FORMULATION

Consider a single-walled carbon nanobeam under the stretching effects and resting on linear and nonlinear elastic foundations (Pasternak, linear and nonlinear Winkler foundations) as shown in Figure 1. Assuming the nanobeam to have homogeneous mass density and cross-sectional area along its length. Also, the nanobeam material is assumed to be isotropic and the mechanical properties of the foundation are uniform along the length of the nanobeam. Based on the assumptions, the governing differential equation are developed as

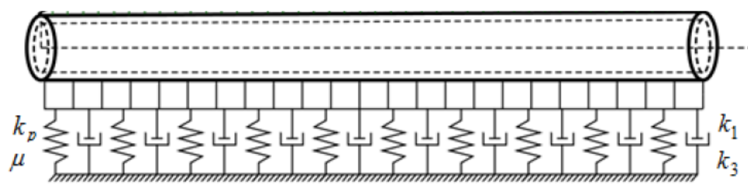


Figure 1 A nanobeam resting on Winkler and Pasternak foundations

$$EI\frac{\partial^4 w}{\partial x^4}+m\frac{\partial^2 w}{\partial t^2}+\mu\frac{\partial w}{\partial t}-k_p\frac{\partial^2 w}{\partial x^2}+k_1 w+k_3 w^3-\left[\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2}=0 \tag{1}$$

subject to the following initial and the boundary conditions (simply-supported nanobeam)

$$w(x,0)=w_o \quad \dot{w}(x,0)=0 \tag{2}$$
$$w(0,t)=w''(0,t)=0 \quad w(L,t)=w''(L,t)=0$$

The derivation of the governing equation is shown in the Appendix.

Using the Galerkin's decomposition procedure to separate the spatial and temporal parts of the lateral displacement functions as

$$w(x,t)=\sum_{n=1}^{N}\phi_n(x)q_n(t) \tag{3}$$

where $q(t)$ the generalized coordinate of the system and $\phi(x)$ is a trial/comparison function that will satisfy both the geometric and natural boundary conditions.

For the simply-supported nanobeam considered in this work

$$\phi(x)=sin\beta_n x \tag{4}$$

where

$$\sin\beta L = 0 \quad \Rightarrow \beta_n = \frac{n\pi}{L} \; n = 1,\, 2,\, 3,\, 4 \ldots .N$$

Therefore, Eq. (3) becomes

$$w(x,t) = \sum_{n=1}^{2} q_n(t)\sin\beta_n x \tag{5}$$

On substituting Eq. (5) into Eq. (1) and apply orthogonal principle of the mode shapes, we arrived at

$$\ddot{q}_1 + \omega_1^2 q_1 + \mu\dot{q}_1 + \alpha_1 q_1^3 + \alpha_3 q_1 q_2^2 = 0 \tag{6a}$$

$$\ddot{q}_2 + \omega_2^2 q_2 + \mu\dot{q}_2 + \alpha_2 q_2^3 + \alpha_4 q_2 q_1^2 = 0 \tag{6b}$$

while for the undamped nanobeam, we have

$$\ddot{q}_1 + \omega_1^2 q_1 + \alpha_1 q_1^3 + \alpha_3 q_1 q_2^2 = 0 \tag{8a}$$

$$\ddot{q}_2 + \omega_2^2 q_2 + \alpha_2 q_2^3 + \alpha_4 q_2 q_1^2 = 0 \tag{8b}$$

where

$$\omega_1^2 = \frac{1}{m}\left\{EI\left(\frac{\pi}{L}\right)^4 + k_p\left(\frac{\pi}{L}\right)^2 + k_1\right\} \; \omega_2^2 = \frac{1}{m}\left\{EI\left(\frac{2\pi}{L}\right)^4 + k_p\left(\frac{2\pi}{L}\right)^2 + k_1\right\}$$

$$\alpha_1 = \frac{1}{m}\left\{\frac{EI}{4}\left(\frac{\pi}{L}\right)^4 + \frac{3}{4}k_3\right\} \; \alpha_3 = \frac{1}{m}\left\{EIA\left(\frac{\pi}{L}\right)^4 + \frac{3}{2}k_3\right\}$$

$$\alpha_2 = \frac{1}{m}\left\{4EIA\left(\frac{\pi}{L}\right)^4 + \frac{3}{4}k_3\right\} \; \alpha_4 = \frac{1}{m}\left\{\frac{EI}{4}\left(\frac{\pi}{L}\right)^4 + \frac{3}{2}k_3\right\}$$

The initial conditions are

$$\begin{aligned} q_1(0) = X_0 \quad \dot{q}_1(0) = 0 \\ q_2(0) = Y_0 \quad \dot{q}_2(0) = 0 \end{aligned} \tag{9}$$

3.0 METHOD OF SOLUTION: VARIATIONAL ITERATION METHOD

In finding direct and practical solutions to the problem, variational iteration method is applied to the simultaneous nonlinear equations. As pointed previously, the variational iteration method is an approximate analytical method for solving differential equations. The basic definitions of the method are as follows

The differential equation to be solved can be written in the form

$$Lu + Nu = g(t) \tag{10}$$

where *L* is a linear operator, *N* is a nonlinear operator and *g(t)* is an inhomogeneous term in the differential equation.

Following VIM procedure, we have a correction functional as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ Lu_n(\tau) + N\tilde{u}(\tau) - g(t) \right\} d\tau \tag{11}$$

λ is a general Lagrange multiplier, the subscript *n* is the nth approximation and $\tilde{u}$ is a restricted variation $\delta\tilde{u} = 0$

Applying the above VIM procedures to Eqs. (7a) and (7b), the following iteration formulations are constructed, letting $u = q_1 \quad v = q_2$

$$u_{n+1} = u_n + \frac{1}{\omega_1}\int_0^t sin\omega_1(\tau - t)\left\{\frac{d^2u_n}{d\tau^2} + \omega_1^2 u_n + \alpha_1 u_n^3 + \alpha_3 u_n v_n^2\right\}d\tau \tag{12a}$$

$$v_{n+1} = v_n + \frac{1}{\omega_2}\int_0^t sin\omega_2(\tau - t)\left\{\frac{d^2v_n}{d\tau^2} + \omega_2^2 v_n + \alpha_2 v_n^3 + \alpha_4 v_n u_n^2\right\}d\tau \tag{12b}$$

In order to find the periodic solution of Eq. (12), we assume an initial approximation for zero-order deformation as

$$u_o = acos(\Omega_1 t) \qquad v_o = bcos(\Omega_2 t) \tag{13}$$

For the first iteration, i.e. *n=0*

$$u_1 = u_0 + \frac{1}{\omega_1}\int_0^t sin\omega_1(\tau - t)\left\{\frac{d^2u_0}{d\tau^2} + \omega_1^2 u_0 + \alpha_1 u_0^3 + \alpha_3 u_0 v_0^2\right\}d\tau \tag{14a}$$

$$v_1 = v_0 + \frac{1}{\omega_2}\int_0^t sin\omega_2\left(\tau - t\right)\left\{\frac{d^2 v_0}{d\tau^2} + \omega_2^2 v_0 + \alpha_2 v_0^3 + \alpha_4 v_0 u_0^2\right\}d\tau \tag{14b}$$

On substituting the corresponding terms in Eq. (13) into Eq. (14a) and (14b), we have

$$u_1 = acos\left(\Omega_1\tau\right) + \frac{1}{\omega_1}\int_0^t sin\omega_1\left(\tau - t\right)\left\{\begin{array}{l} a\left(\omega_1^2 - \Omega_1^2\right)cos\left(\Omega_1\tau\right) + \alpha_1 a^3 cos^3\left(\Omega_1\tau\right) \\ +\alpha_3 ab^2 cos\left(\Omega_1\tau\right)cos^2\left(\Omega_1\tau\right)\end{array}\right\}d\tau \tag{15a}$$

$$v_1 = bcos\left(\Omega_1\tau\right) + \frac{1}{\omega_2}\int_0^t sin\omega_2\left(\tau - t\right)\left\{\begin{array}{l} b\left(\omega_2^2 - \Omega_2^2\right)cos\left(\Omega_2\tau\right) + \alpha_2 b^3 cos^3\left(\Omega_2\tau\right) \\ +\alpha_4 b^2 Acos\left(\Omega_2\tau\right)cos^2\left(\Omega_2\tau\right)\end{array}\right\}d\tau \tag{15b}$$

It should be pointed out that Ω_1 and Ω_2 are the nonlinear natural frequencies.
After mathematical calculations and simplifications of Eq. (15a) and (15b), we have

$$\begin{aligned} u_1 = & \left\{\begin{array}{l} a + \dfrac{\alpha_3 ab^2}{2\left(\omega_1^2 - \Omega_1^2\right)} + \dfrac{\alpha_3 ab^2}{4}\left[\dfrac{1}{\left[\omega_1^2 - \left(2\Omega_2 + \Omega_1\right)^2\right]} + \dfrac{1}{\left[\omega_1^2 - \left(2\Omega_2 - \Omega_1\right)^2\right]}\right] \\ + \dfrac{\alpha_1 a^3}{4}\left[\dfrac{3}{\omega_1^2 - \Omega_1^2} + \dfrac{1}{\omega_1^2 - 9\Omega_1^2}\right] \end{array}\right\} cos\left(\Omega_1 t\right) \\ & - \frac{\alpha_3 ab^2}{4}\left[\frac{2cos\left(\Omega_1 t\right)}{\omega_1^2 - \Omega_1^2} + \frac{cos\left[\left(2\Omega_1 + \Omega_1\right)t\right]}{\left[\omega_1^2 - \left(2\Omega_2 + \Omega_1\right)^2\right]} + \frac{2cos\left[\left(2\Omega_1 - \Omega_1\right)t\right]}{\left[\omega_1^2 - \left(2\Omega_2 - \Omega_1\right)^2\right]}\right] \\ & - \frac{\alpha_1 a^3}{4}\left[\frac{3cos\left(\Omega_1 t\right)}{\omega_1^2 - \Omega_1^2}\right] + \frac{cos\left(3\Omega_1 t\right)}{\omega_1^2 - 9\Omega_1^2} \end{aligned} \tag{16a}$$

$$v_1=\begin{Bmatrix}b+\frac{\alpha_4 ba^2}{2\left(\omega_2^2-\Omega_2^2\right)}+\frac{\alpha_4 ba^2}{4}\left[\frac{1}{\left[\omega_2^2-\left(2\Omega_1+\Omega_2\right)^2\right]}+\frac{1}{\left[\omega_1^2-\left(2\Omega_2-\Omega_2\right)^2\right]}\right]\\+\frac{\alpha_2 b^3}{4}\left[\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\right]\end{Bmatrix}cos\left(\Omega_2 t\right)$$
$$-\frac{\alpha_4 ba^2}{4}\left[\frac{2cos\left(\Omega_2 t\right)}{\omega_2^2-\Omega_2^2}+\frac{cos\left[\left(2\Omega_1+\Omega_2\right)t\right]}{\left[\omega_2^2-\left(2\Omega_1+\Omega_2\right)^2\right]}+\frac{2cos\left[\left(2\Omega_1-\Omega_2\right)t\right]}{\left[\omega_1^2-\left(2\Omega_1-\Omega_2\right)^2\right]}\right]$$
$$-\frac{\alpha_2 b^3}{4}\left[\frac{3cos\left(\Omega_2 t\right)}{\omega_2^2-\Omega_2^2}\right]+\frac{cos\left(3\Omega_2 t\right)}{\omega_2^2-9\Omega_2^2} \tag{16b}$$

We should recall from Eq. (5),

$$w(x,t)=\sum_{n=1}^{2}q_n(t)sin\beta_n x=q_1(t)sin\beta_1 x+q_2(t)sin\beta_2 x=u_1 sin\beta_1 x+v_1 sin\beta_2 x \tag{17}$$

Therefore,

$$w(x,t)\approx\begin{Bmatrix}\begin{Bmatrix}a+\frac{\alpha_3 ab^2}{2\left(\omega_1^2-\Omega_1^2\right)}+\frac{\alpha_3 ab^2}{4}\left[\frac{1}{\left[\omega_1^2-\left(2\Omega_2+\Omega_1\right)^2\right]}+\frac{1}{\left[\omega_1^2-\left(2\Omega_2-\Omega_1\right)^2\right]}\right]\\+\frac{\alpha_1 a^3}{4}\left[\frac{3}{\omega_1^2-\Omega_1^2}+\frac{1}{\omega_1^2-9\Omega_1^2}\right]\end{Bmatrix}cos\left(\Omega_1 t\right)\\-\frac{\alpha_3 ab^2}{4}\left[\frac{2cos\left(\Omega_1 t\right)}{\omega_1^2-\Omega_1^2}+\frac{cos\left[\left(2\Omega_1+\Omega_1\right)t\right]}{\left[\omega_1^2-\left(2\Omega_2+\Omega_1\right)^2\right]}+\frac{2cos\left[\left(2\Omega_1-\Omega_1\right)t\right]}{\left[\omega_1^2-\left(2\Omega_2-\Omega_1\right)^2\right]}\right]-\frac{\alpha_1 a^3}{4}\left[\frac{3cos\left(\Omega_1 t\right)}{\omega_1^2-\Omega_1^2}\right]+\frac{cos\left(3\Omega_1 t\right)}{\omega_1^2-9\Omega_1^2}\end{Bmatrix}sin\beta_1 x$$
$$+\begin{Bmatrix}\begin{Bmatrix}b+\frac{\alpha_4 ba^2}{2\left(\omega_2^2-\Omega_2^2\right)}+\frac{\alpha_4 ba^2}{4}\left[\frac{1}{\left[\omega_2^2-\left(2\Omega_1+\Omega_2\right)^2\right]}+\frac{1}{\left[\omega_1^2-\left(2\Omega_2-\Omega_2\right)^2\right]}\right]\\+\frac{\alpha_2 b^3}{4}\left[\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\right]\end{Bmatrix}cos\left(\Omega_2 t\right)\\-\frac{\alpha_4 ba^2}{4}\left[\frac{2cos\left(\Omega_2 t\right)}{\omega_2^2-\Omega_2^2}+\frac{cos\left[\left(2\Omega_1+\Omega_2\right)t\right]}{\left[\omega_2^2-\left(2\Omega_1+\Omega_2\right)^2\right]}+\frac{2cos\left[\left(2\Omega_1-\Omega_2\right)t\right]}{\left[\omega_1^2-\left(2\Omega_1-\Omega_2\right)^2\right]}\right]-\frac{\alpha_2 b^3}{4}\left[\frac{3cos\left(\Omega_2 t\right)}{\omega_2^2-\Omega_2^2}\right]+\frac{cos\left(3\Omega_2 t\right)}{\omega_2^2-9\Omega_2^2}\end{Bmatrix}sin\beta_2 x \tag{18}$$

4.0 DETERMINATION OF NATURAL FREQUENCY OF THE VIBRATION

In order to find the natural frequency of the vibration, we have to eliminate the secular term. After eliminating the secular term in *u* and *v*, we have

$$a+\frac{\alpha_3 ab^2}{2\left(\omega_1^2-\Omega_1^2\right)}+\frac{\alpha_3 ab^2}{4}\left[\frac{1}{\left[\omega_1^2-\left(2\Omega_2+\Omega_1\right)^2\right]}+\frac{1}{\left[\omega_1^2-\left(2\Omega_2-\Omega_1\right)^2\right]}\right]+\frac{\alpha_1 a^3}{4}\left[\frac{3}{\omega_1^2-\Omega_1^2}+\frac{1}{\omega_1^2-9\Omega_1^2}\right]=0 \tag{19a}$$

And

$$b+\frac{\alpha_4 ba^2}{2\left(\omega_2^2-\Omega_2^2\right)}+\frac{\alpha_4 ba^2}{4}\left[\frac{1}{\left[\omega_2^2-\left(2\Omega_1+\Omega_2\right)^2\right]}+\frac{1}{\left[\omega_1^2-\left(2\Omega_2-\Omega_2\right)^2\right]}\right]+\frac{\alpha_2 b^3}{4}\left[\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\right]=0 \tag{19b}$$

It should be noted that from Eq. (3), Eq. (5) and the initial conditions in Eq. (9) that

$$w(x,0)=u_o sin\beta_1 x+v_o sin\beta_2 x=w_o \tag{20}$$

Which can be written as

$$acos(\Omega_1 t)sin\beta_1 x+bcos(\Omega_2 t)sin\beta_2 x=w_o \tag{21}$$

For the general case of $a,b\neq 0$, Eq. (19a) and Eq. (19b) implicitly generate the main frequencies of the symmetric and un-symmetric modes of oscillations. However, for the special case of $a\neq 0,\quad b=0$, after simplifications, we arrived at

$$9\Omega_1^4-\left(10\omega_1^2+7\alpha_1 a^2\right)\Omega_1^2+\omega_1^2\left(\omega_1^2+\alpha_1 a^2\right)=0 \tag{22}$$

On solving Eq. (22), we arrived at

$$\Omega_1 = \sqrt{\frac{\left(10\omega_1^2 + 7\alpha_1 a^2\right) + \sqrt{\left(10\omega_1^2 + 7\alpha_1 a^2\right)^2 - 36\omega_1^2\left(\omega_1^2 + \alpha_1 a^2\right)}}{18}} \tag{23}$$

A further simplification gives

$$\Omega_1 = \sqrt{\frac{\left(10\omega_1^2 + 7\alpha_1 a^2\right) + \sqrt{64\omega_1^4 + 104\omega_1^2\alpha_1 a^2 + 49\alpha_1^2 a^4}}{18}} \tag{24}$$

And

$$u_1 = \left\{a + \frac{\alpha_1 a^3}{4}\left[\frac{3}{\omega_1^2 - \Omega_1^2} + \frac{1}{\omega_1^2 - 9\Omega_1^2}\right]\right\} cos\left(\Omega_1 t\right) - \frac{\alpha_1 a^3}{4}\left[\frac{3cos\left(\Omega_1 t\right)}{\omega_1^2 - \Omega_1^2}\right] + \frac{cos\left(3\Omega_1 t\right)}{\omega_1^2 - 9\Omega_1^2} \tag{25a}$$

$$v_1 = 0 \tag{25b}$$

Therefore,

$$w(x,t) \approx \left\{\left\{a + \frac{\alpha_1 a^3}{4}\left[\frac{3}{\omega_1^2 - \Omega_1^2} + \frac{1}{\omega_1^2 - 9\Omega_1^2}\right]\right\} cos\left(\Omega_1 t\right) - \frac{\alpha_1 a^3}{4}\left[\frac{3cos\left(\Omega_1 t\right)}{\omega_1^2 - \Omega_1^2}\right] + \frac{cos\left(3\Omega_1 t\right)}{\omega_1^2 - 9\Omega_1^2}\right\} sin\beta_1 x$$

(26)

where

$$a = w_o sec\,(\Omega_1 t) cosec \beta_1 x$$

Also, for the special case of $a = 0, \quad b \neq 0$, after simplifications, we arrived at

$$9\Omega_2^4 - \left(10\omega_2^2 + 7\alpha_2 b^2\right)\Omega_2^2 + \omega_2^2\left(\omega_2^2 + \alpha_2 b^2\right) = 0 \tag{27}$$

Which gives

$$\Omega_2 = \sqrt{\frac{\left(10\omega_2^2 + 7\alpha_2 b^2\right) + \sqrt{\left(10\omega_2^2 + 7\alpha_2 b^2\right)^2 - 36\omega_2^2\left(\omega_2^2 + \alpha_2 b^2\right)}}{18}} \tag{28}$$

A further simplification gives

$$\Omega_2 = \sqrt{\frac{\left(10\omega_2^2 + 7\alpha_2 b^2\right) + \sqrt{64\omega_2^4 + 104\omega_2^2\alpha_2 b^2 + 49\alpha_2^2 b^4}}{18}} \tag{29}$$

And

$$u_1 = 0 \tag{30a}$$

$$v_1 = \left\{b + \frac{\alpha_2 b^3}{4}\left[\frac{3}{\omega_2^2 - \Omega_2^2} + \frac{1}{\omega_2^2 - 9\Omega_2^2}\right]\right\} cos\left(\Omega_2 t\right) - \frac{\alpha_2 b^3}{4}\left[\frac{3cos\left(\Omega_2 t\right)}{\omega_2^2 - \Omega_2^2}\right] + \frac{cos\left(3\Omega_2 t\right)}{\omega_2^2 - 9\Omega_2^2} \tag{30b}$$

Therefore

$$w(x,t) \approx \left\{\left\{b + \frac{\alpha_2 b^3}{4}\left[\frac{3}{\omega_2^2 - \Omega_2^2} + \frac{1}{\omega_2^2 - 9\Omega_2^2}\right]\right\} cos\left(\Omega_2 t\right) - \frac{\alpha_2 b^3}{4}\left[\frac{3cos\left(\Omega_2 t\right)}{\omega_2^2 - \Omega_2^2}\right] + \frac{cos\left(3\Omega_2 t\right)}{\omega_2^2 - 9\Omega_2^2}\right\} sin\beta_2 x$$

(31)

where

$$b = w_o sec(\Omega_2 t) cosec\beta_2 x$$

Also, it can easily be seen that as the nonlinear term tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\Omega_{1,2}}{\omega_{1,2}}$ tends to 1.

$$\lim_{\alpha_{1,2,3,4} \to 0} \frac{\Omega_{1,2}}{\omega_{1,2}} = 1$$

(32)

Also, as the amplitudes a and b tend to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\Omega_{1,2}}{\omega_{1,2}}$ tends to 1.

$$\lim_{a,b \to 0} \frac{\Omega_{1,2}}{\omega_{1,2}} = 1$$

(33)

For very large values of the amplitudes *a*, *b*, we have

$$\lim_{a,b\to\infty} \frac{\Omega_{1,2}}{\omega_{1,2}} = \infty$$
(34)

Table: Parameters used for the simulation

S/N	Parameter	Symbol	Values used
1.	Modulus of elasticity	E	1-1.2×10^{12} Pa
2.	Density of the nanobeam	ρ	1.2-2.3 ×10^3 kg/m^3
3.	Winkler foundation constant,	k_1	0-10^6 N/m^2
4.	Pasternak linear foundation constant,	k_p	0-10^{-5} N/m^2
5.	Pasternak nonlinear foundation constant	k_3	0-10^{15}N/m^2
6.	Length of the nanobeam	L	10-100 nm
7.	Diameter of the nanobeam	d	0.5- 6 nm

5.0 RESULTS AND DISCUSSIONS

The first-five normalized mode shapes of the simple-simple beam are shown in Figure 2. Also, the figure shows the deflections of the beam along the beams' span at five different buckled and mode shapes. From the first mode shape, the highest deflection occurs at the mid-span of the beam due to the symmetrical nature of the boundary conditions of the simply-simply support beam.

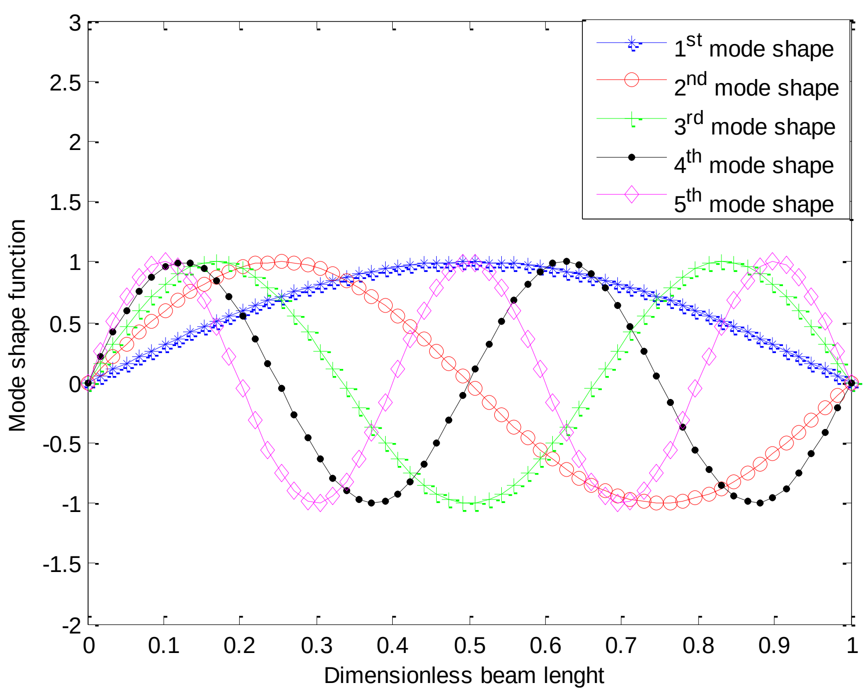


Figure 2. The first five normalized mode shaped of the

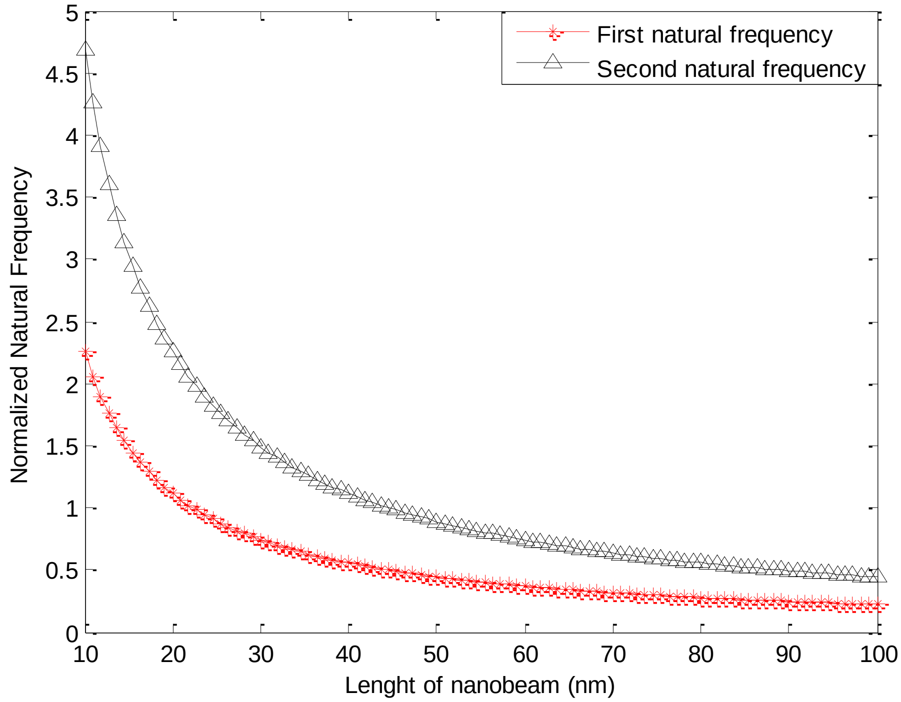


Figure 3. Variation of nanotube length on natural frequency under simple-simple supports

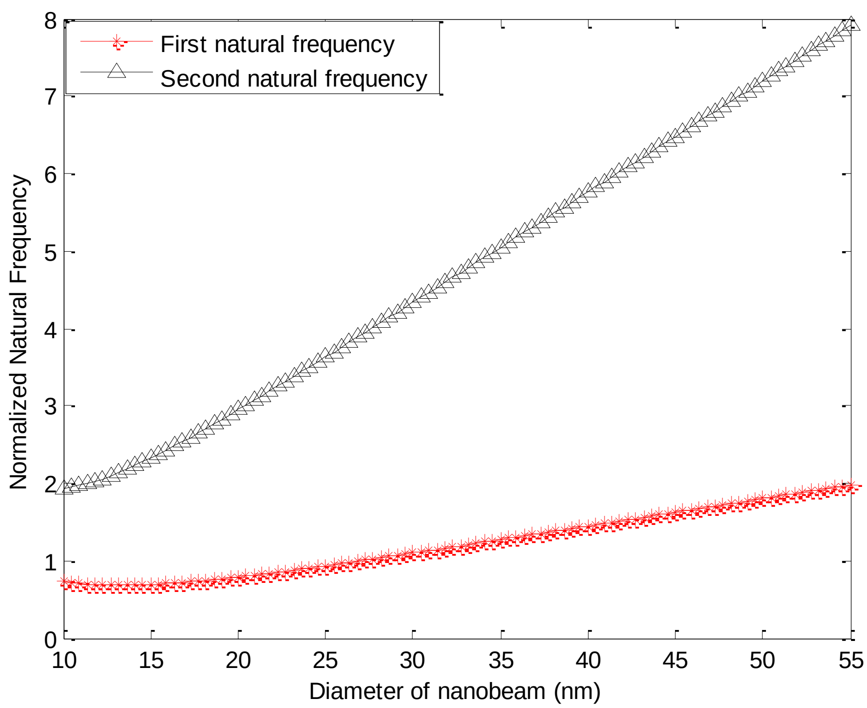


Figure 4. Variation of nanotube length on natural frequency

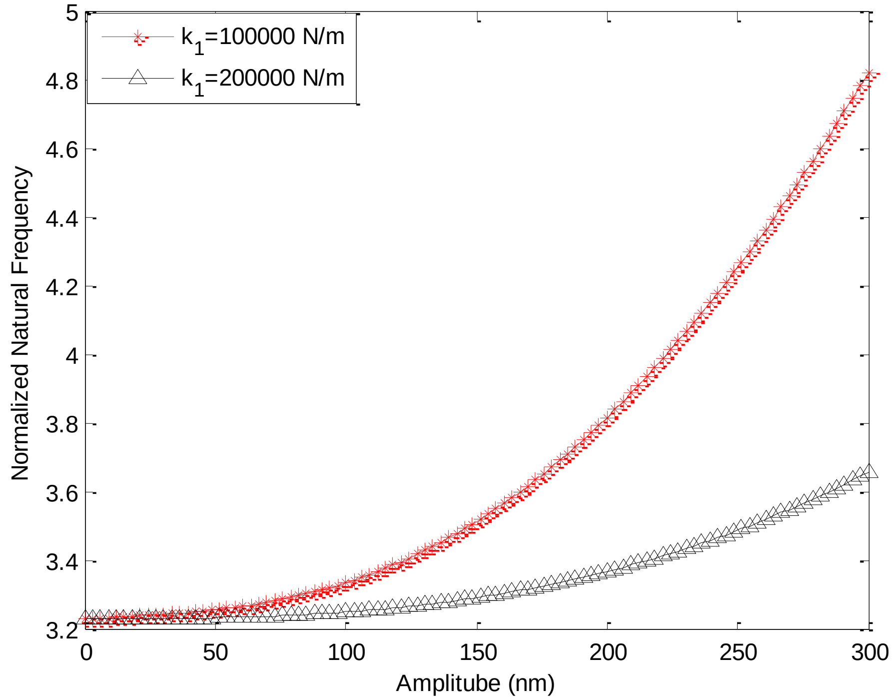


Figure 5. Effects of Winkler foundation parameter on natural frequency

Figure 3 and 4 show the effects of nanobeam length and diameter on the normalized natural frequencies of the beam, respectively. The increase in the beam length decreases the natural frequency of vibration while as the diameter of the nanobeam increases, the natural frequency increases. The observations are in good agreements with the established results in literature. Figure 5 is associated with the variation of the nonlinear frequency ratios of the CNB with spring constant with spring constant of the foundation/surrounding medium. From the figure, it could be seen that with the increase of the spring constant, the nonlinear frequency ratio decreases. It is observed that by increasing the spring constant of the surrounding medium, the nonlinear frequencies get close to the linear frequencies so that nonlinearity becomes less evident for the spring constants of large enough i.e. the influence of nonlinearity is more prominent for low stiffness of elastic media. However, for high stiffness media, nonlinear vibration frequencies are very close to linear ones. This establishes that at a high stiffness media, the CNB behavior can be modeled as a linear system whose geometric nonlinearity becomes negligible. Also, the variation of nonlinear frequency with the non-dimensional amplitude for CNB is depicted in the figure. In contrast to linear systems, the nonlinear frequency ratio is strongly dependent on amplitude so that the larger the amplitude, the more pronounced the discrepancy between the linear and nonlinear frequencies becomes. This mean that the nonlinear frequency of nanobeam increases with increase in the vibration amplitude. Also, it was found that the discrepancy between the linear and nonlinear responses tends to increase as time evolves. Figure 6 depicts that as the foundation parameter increases, the nonlinear vibration frequency ratio increases and the difference between the nonlinear and linear frequency becomes pronounced

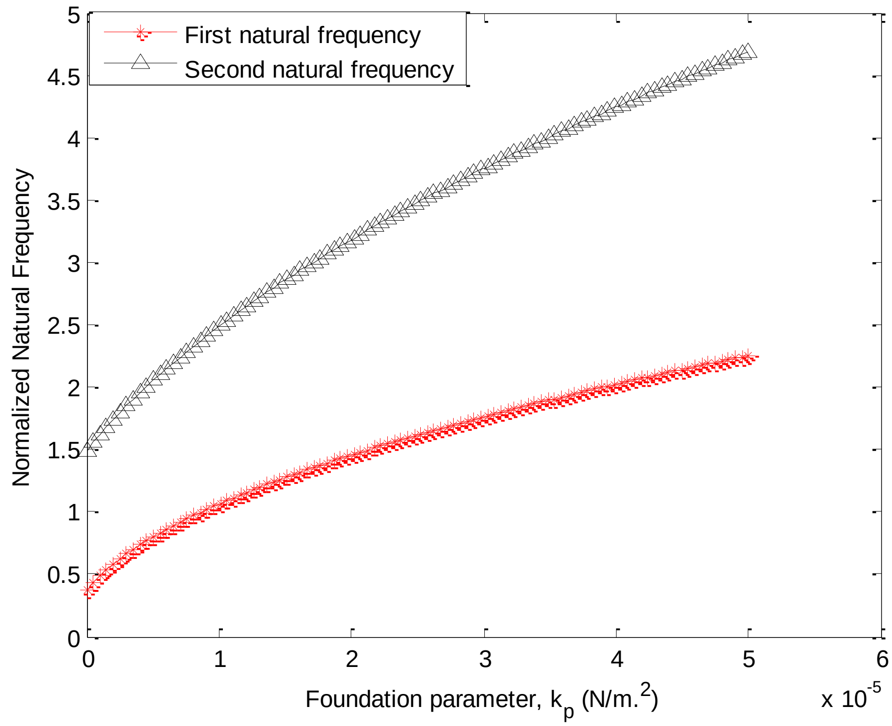
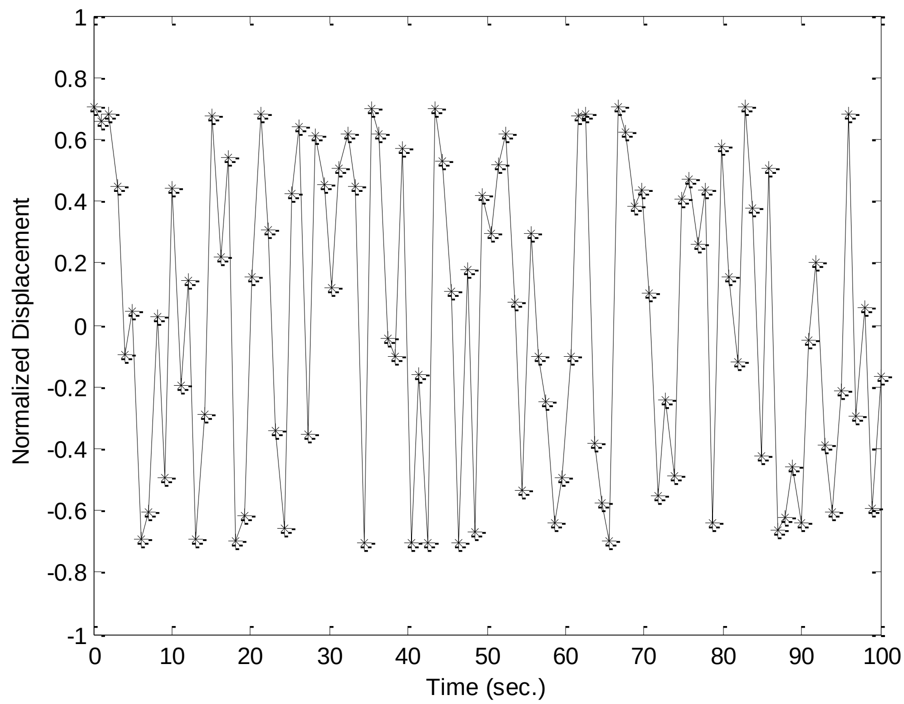


Figure 6. Effects of foundation parameter on natural frequency



.

Figure 7. Effects of foundation parameter on the midpoint displacement

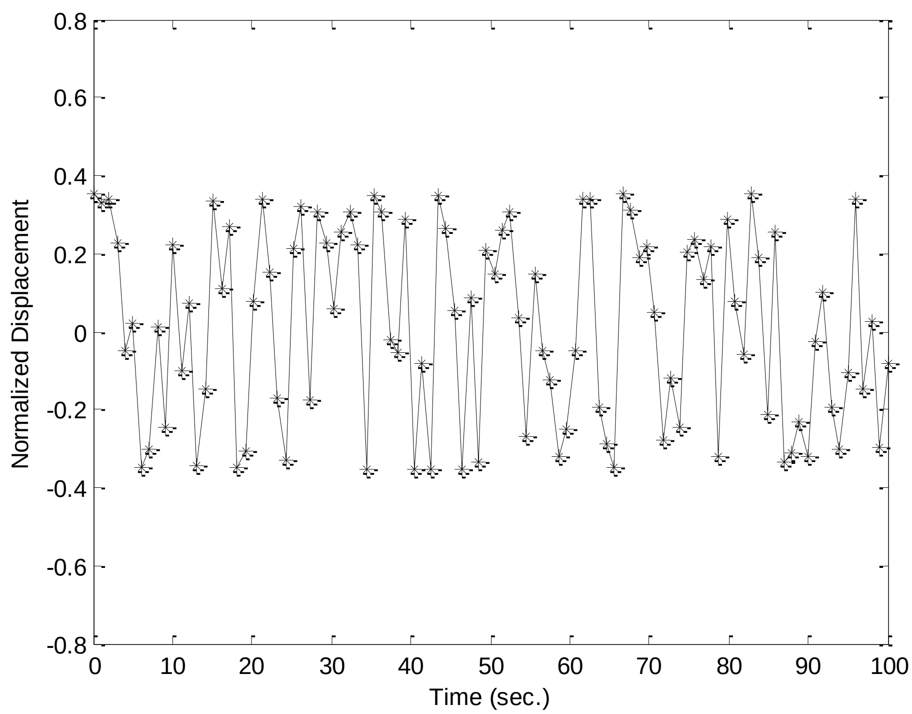


Figure 8. Effects of foundation parameter on midpoint displacement

Effects of foundation parameter on the midpoint deflection time history are illustrated in Figure 7 and 8 . Figure 7 displays the midpoint deflection time history for the nonlinear analysis of carbon nanobeam when $k_p=0.01$ while Figure 8 presents the midpoint deflection time history for the nonlinear analysis of carbon nanobeam when $k_p = 0.03$

6.0 CONCLUSIONS

In this work, nonlinear vibration analysis of nanobeam has been studied under the effects of stretching and Winkler and Pasternak foundations using variational iteration method. The increase in the beam length decreases the natural frequency of vibration while as the diameter of the nanobeam increases, the natural frequency increase. The increase of the spring constant, the nonlinear frequency ratio decreases. It was established that at a high stiffness media, the CNB behavior can be modeled as a linear system whose geometric nonlinearity becomes negligible. The nonlinear frequency of nanobeam increases with increase in the vibration amplitude and the discrepancy between the linear and nonlinear responses tends to increase when time evolved. As the foundation parameter increases, the nonlinear vibration frequency ratio increases and the difference between the nonlinear and linear frequency becomes pronounced. These analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities in the problems as it provides clearer insights to understanding the problems in comparison with numerical methods.

7.0 REFERENCES

Chen, C. K. and S.H. Ho, S. H. (1996). Application of Differential Transformation to Eigenvalue Problems. *Journal of Applied Mathematics and Computation*, 79, 173-188.

El-Shahed, M. (2008). Application of differential transform method to non-linear oscillatory systems. *Communic. Nonlin. Scien. Numer. Simul.* 13, 1714-1720.

Fernandez. A. (2009). On some approximate methods for nonlinear models. *Appl Math Comput*., 215. :168-74.

Ganji, S. S., Ganji, D. D., Babazadeh, H. and Karimpour, S. (2008). Variational approach method for nonlinear oscillations of the motion of a rigid rod rocking back and cubic-quinticduffing oscillators. *Prog. Electromagn*. Res. M 4, 23–32.

He J. H. (1998a). Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Computer Methods in Applied Mechanics and Engineering* 167 (1–2), 57–68.

He J. H. (1998b)., Approximate solution of nonlinear differential equations with convolution product nonlinearities, *Computer Methods in Applied Mechanics and Engineering* 167 (1–2), 69–73.

He J. H. (1999). Variational iteration method — a kind of non-linear analytical technique: Some examples, *International Journal of Non-Linear Mechanics* 34 (4), 699–708.

He J. H. (2000). Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation* 118 (2–3) ,115–123.

He J. H. (2006). X.H.Wu, Construction of solitary solution and compacton-like solution by variational iteration method, *Chaos*, *Solitons and Fractals*, 29 (1), 108–113.

He J. H. (2007a). Variational iteration method—some recent results and new interpretations, *Journal of Computational and Applied Mathematics*, 207 (1), 3–17.

He J. H. and Wu X. H. (2007b). Variational iteration method: New development and applications, *Comput. Meth. Appl. Mech. Eng*, 54(7-8): 881-894.

He J. H. (2011). A short remark on fractional variational iteration method, *Phys. Lett. A*, 375(38) 3362-3364

He J. H. (2012a). An Approximation to Solution of Space and Time Fractional Telegraph Equations by the Variational Iteration Method , *Mathematical Problems in Engineering*, 394212

He J. H. (2012b). Comment on "Variational Iteration Method for Fractional Calculus Using He's Polynomials" , *Abstract and Applied Analysis*, 964-974

Hesameddini E, Latifizadeh H. (2009). Reconstruction of Variational Iteration Algorithms using the Laplace Transform, *Int. J. Nonlin. Sci. Num*., 10(11-12): 1377-1382

Liao, S. J. (1992) The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph. D. dissertation, *Shanghai Jiao Tong University*.

Liao, S. J. (1995) An approximate solution technique not depending on small parameters: a special example. *Int. J. Non-Linear Mech.* 30(3), 371–380.

Liao, S. J. and Tan, Y. A. (2007). General approach to obtain series solutions of nonlinear differential equations, *Studies Appl. Math.* 119(4), 297–354.

Marinca, V. and Herisanu, N. (2006). A modified iteration perturbation method for some nonlinear oscillation problems. *Acta Mech*. 184, 231–242.

Momani, S. S. (2004). Analytical approximate solutions of non-linear oscillators by the modified decomposition method. *Int. J. Modern. Phys. C*, 15(7): 967-979.

Rafei., M., Ganji, D. D., Daniali, H. and Pashaei, H. (2007). The variational iteration method for nonlinear oscillators with discontinuities. *J. Sound Vib.* 305, 614–620.

Rafiepour, H., Tabatabaei, S. H. and Abbaspour. M. (2014) A novel approximate analytical method for nonlinear vibration analysis of Euler-Bernoulli and Rayleigh beams on the nonlinear foundation. *Arab J. Sci Eng*.

Sobamowo, M. G. (2016). Thermal analysis of longitudinal fin with temperature-dependent properties and internal heat generation using Galerkin's method of weighted residual. *Applied Thermal Engineering* 99 (2016) 1316–1330.

Wu G. C. (2012). Laplace transform overcoming principal drawbacks in application of the variational iteration method to fractional heat equations, *Thermal Science*, 16(4): 1257-1261

Zhou, J. K. (1986) Differential Transformation and its Applications for Electrical Circuits. *Huazhong University Press: Wuhan, China.*

NOMENCLATURE

A Area of the structure
E Young Modulus of Elasticity
I moment of area
k_1, k_2, k_3 foundation constants
L length of the nanobeam
m_p mass of the nanobeam
N axial/Longitudinal force
t time
$u(t)$ generalized coordinate of the system
w transverse displacement/deflection
x axial coordinate
σ_v tangential moment accommodation coefficient
$\phi(x)$ trial/comparison function

APPENDIX

Using Euler-Bernoulli theory,the governing equation of motion as derived as follows.

The bending moment for the Euler-Bernoulli beam is given as

$$M(x,t) = \int_A z\sigma_{xx} dA$$

(A1)

where A, z, σ_{xx} is the cross sectional area of the nanotube, distance from the neutral axis and the axial stress on the nanotube, respectively

It should be noted that

$$\sigma_{xx} = E\varepsilon_{xx}$$

(A2)

ε_{xx} is the axial strain of the nanotube

On substituting Eq. (A1) into Eq. (A2), we have

$$M(x,t) = \int_A zE\varepsilon_{xx} dA$$

(A3)

Following von Karman strain, we have

$$\varepsilon_{xx} \approx -z\frac{\partial^2 w}{\partial x^2}$$

(A4)

Where *w*is the displacement of the nanotube

On substituting Eq. (A4) into Eq. (A3), we have

$$M(x,t) = -E\frac{\partial^2 w}{\partial x^2}\int_A z^2 dA$$

(A5)

But the second moment of area,

$$I = \int_A z^2 dA$$

(A6)

Therefore,

$$M(x,t) = -EI\frac{\partial^2 w}{\partial x^2}$$

(A7)

By incorporating von Karman's nonlinearity, the internal shear force on the structural cross section must satisfy themoment equilibrium relation

$$V(x,t) = \frac{\partial M}{\partial x} + N(x,t)\frac{\partial w}{\partial x}$$

(A8)

It should be pointed out that the internal membrane force, N is constant along the beam as

$$\frac{\partial N}{\partial x} = 0 \quad \Rightarrow N(x,t) = N(t)$$
(A9)
Therefore, Eq. (A9) becomes

$$V(x,t) = \frac{\partial M}{\partial x} + N(t)\frac{\partial w}{\partial x}$$
(A10)
Differentiating Eq. (A10) with respect to spatial variable x considering the absence of external axial load on the beam

$$\frac{\partial^2 M}{\partial x^2} = \frac{\partial V}{\partial x} - N(t)\frac{\partial^2 w}{\partial x^2}$$
(A11)
Using Newton's law, the governing equation of motion for the free vibration of the nanotube can be expressed as

$$\frac{\partial V}{\partial x} = m\frac{\partial^2 w}{\partial t^2} + \mu\frac{\partial w}{\partial t} - k_p\frac{\partial^2 w}{\partial x^2} + k_1 w + k_3 w^3$$
(A12)

Substituting Eq. (A12) into Eq. (A11), we have

$$\frac{\partial^2 M}{\partial x^2} = m\frac{\partial^2 w}{\partial t^2} + \mu\frac{\partial w}{\partial t} - k_p\frac{\partial^2 w}{\partial x^2} + k_1 w + k_3 w^3 - N(t)\frac{\partial^2 w}{\partial x^2}$$
(A13)

For the immovable supports, the internal membrane force is given as

$$N(t) = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx$$
(A14)

Therefore, Eq. (A13) can be expressed

$$\frac{\partial^2 M}{\partial x^2} = m\frac{\partial^2 w}{\partial t^2} + \mu\frac{\partial w}{\partial t} - k_p\frac{\partial^2 w}{\partial x^2} + k_1 w + k_3 w^3 - \left[\frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2}$$

(A15)
Where k_1, k_3 and k_p are the Pasternak, linear and nonlinear Winkler foundation constants

From Equ. (A7), we have

$$\frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^4 w}{\partial x^4}$$
(A16)

If we substitute Eq. (A16) into Eq. (A15), we obtained the governing equation as motion for the nanotube as

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} + \mu\frac{\partial w}{\partial t} - k_p\frac{\partial^2 w}{\partial x^2} + k_1 w + k_3 w^3 - \left[\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} = 0$$
(A17)