THERMAL PERFORMANCE AND OPTIMUM DESIGN ANALYSIS OF FIN WITH VARIABLE THERMAL CONDUCTIVITY USING DOUBLE DECOMPOSITION METHOD

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ABSTRACT

In this paper, thermal performance study and optimum design analysis of straight fin with variable thermal conductivity are carried out using double decomposition method. The developed heat transfer models are used to analyze the thermal performance, establish the optimum thermal design parameters and also, investigate the effects of thermo-geometric parameters and thermal conductivity (non-linear) parameters on the temperature distribution, heat transfer and thermal performance of the longitudinal rectangular fin. From the results, it is established that the fin temperature distribution, the total heat transfer, the fin effectiveness, and the fin efficiency are significantly affected by the thermo-geometric and thermal parameters of the fin. Also, it is established that the optimum fin length increases as the non-linear thermal conductivity term, increases. Therefore, the operational parameters must be carefully chosen to ensure that the fin retains its primary purpose of removing heat from the primary surface. The results obtained in this analysis provides platform for improvement in the design of fin in heat transfer equipment.

KEYWORDS**:** *Performance analysis; Convective Optimal design; Longitudinal Fin; Double decomposition method; Temperature-dependent thermal conductivity.*

1.0 INTRODUCTION

High-performance heat transfer components with progressively small weights, volume and costs are continuously demanded in large numbers of thermal systems. Consequently, fins are widely employed in the design and construction of various types of heat-transfer equipment and components such as air conditioning, refrigeration, superheaters, automobile, power plants, heat exchangers, convectional furnaces, economizers, gas turbines, chemical processing equipment, oil carrying pipelines, computer processors, electrical chips etc. The extended surfaces are used to increase the rate of heat transfer

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between the primary surface and the fin. In practice, various types of fins with different geometries are used, but due to simplicity of its design and ease of construction and manufacturing process, the rectangular fins are widely used. Also, for ordinary fins problem, the thermal properties of the fin thermal conductivity is assumed to be constant, but if large temperature difference exists within the fin, typically, between tip and the base of the fin (such as heat pipe, space radiator etc.), the thermal conductivity is temperature-dependent. These facts attest that for many engineering applications, the thermal conductivity is temperature-dependent. Therefore, while analyzing the fin under such situations, effects of the temperature-dependent thermal properties must be taken into consideration. In carrying out such analysis, the thermal conductivity may be modelled for such and other many engineering applications by power law and by linear dependency on temperature (Khani & Aziz, 2010; Ndlovu & Moitsheki, 2013). Such dependency of thermal conductivity renders the problem non-linear and difficult to solve exactly. Over the past few decades, the solution of the governing non-linear differential equations has been constructed using different techniques. Aziz and Enamul-Huq (1973) applied regular perturbation expansion to study a pure convection fin with temperature dependent thermal conductivity. Aziz (1977) extended the previous analysis to include a uniform internal heat generation in the fin. Few years later, Campo and Spaulding (1999) applied method of successive approximation to predict the thermal behaviour of uniform circumferential fins.

Chiu and Chen (2002) and Arslanturk (2005) adopted the Adomian decomposition Method (ADM) to obtain the temperature distribution in a pure convection fin with variable thermal conductivity. The same problem was also solved by Ganji (2006) with the aid of the homotopy perturbation method originally proposed by He (1999). Chowdhury and Hashim (2008) applied the Adomian decomposition method to evaluate the temperature distribution of straight rectangular fin with temperature dependent surface flux for all possible types of heat transfer. In the following year, Rajabi (2007) employed Homotopy perturbation method (HPM) to calculate the efficiency of straight fins with temperature-dependent thermal conductivity. A year later, Mustapha (2008) adopted Homotopy analysis method (HAM) to find the efficiency of straight fins with temperature-dependent thermal conductivity. Also, Coskun and Atay (2007) utilized variational iteration method (VIM) for the analysis of convective straight and radial fins with temperature-dependent thermal conductivity while Languri et al. (2008) applied both variation iteration and Homotopy perturbation methods for the evaluation of efficiency of straight fins with temperature-dependent thermal conductivity. Coskun and Atay (2008) applied variational iteration method to analyse the efficiency of convective straight fins with temperature-dependent thermal conductivity. In the same year, Atay and Coskum (2008) employed variation iteration and finite element methods to carry out comparative analysis of power-law-fin type problems. Domairry and Fazeli (2009) used Homotopy analysis method to determine the efficiency of straight fins with temperature-dependent thermal conductivity.

Chowdhury et al.(2009) investigated a rectangular fin with power law surface heat flux and made a comparative assessment of results predicted by HAM, HPM and ADM. Khani *et al.* (2009) used Adomian decomposition method (ADM) to provide series solution to fin problem with a temperature-dependent thermal conductivity. Moitsheki et al. (2010)

applied the Lie symmetry analysis to provide exact solutions of the fin problem with a power-law temperature-dependent thermal conductivity. Also, in the same year, Hosseini et al. (2012) applied homotopy analysis method to provide approximate but accurate solution of heat transfer in fin with temperature-dependent internal heat generation and thermal conductivity. To the best of the authors' knowledge, very few studies were actually directed to the analysis of heat transfer in fins with temperature-dependent thermal properties while the study of fin with temperature-dependent internal heat generation, thermal conductivity and heat transfer coefficient are very limited or scarcely carried out in literature. Furthermore, differential transform method (DTM) solves the differential equations without linearization, discretization or no approximation, linearization restrictive assumptions or perturbation, complexity of expansion of derivatives and computation of derivatives symbolically. This method was applied by Joneidi et al. (2009), Moradi and Ahmadikia (2010) as well as Moradi (2010) presented analytical solution for fin with temperature dependent thermal coefficient.

The method was also used by Mosayebidorcheh et al. (2014), Ghasemi et al. (2014), Sandri et al. (2012), Ganji and Dogonchi (2014) also applied the DTM to solve the fin problem but the search for an auxiliary value that will satisfy the second boundary condition necessitated the use of Maple software and such results in additional computational costs and efforts in the generation of solution to the problem. This drawback is not only peculiar to DTM, other approximate analytical methods such as HPM, HAM, ADM and VIM also required additional computational cost, time and efforts for the determination of auxiliary parameters which could lead to tedious and very complicated work to do. Also, DTM only provides acceptable approximation for small range i.e. it does not exhibit a good approximation in large domain. This is because a boundary condition is satisfied via the method, and the remaining unsatisfied boundary condition plays no roles in the final results. This deficiency limits the efficiency and the applications of DTM over wide range of problems. HPM, HAM, ADM and VIM often involved complex mathematical analysis leading to analytic expression involving a large number terms and when such methods are routinely implemented, they can sometimes lead to erroneous results (Fernandez, 2009) and (Aziz and Bouaziz, 2011). In practice, approximate analytical solutions with large number of terms are not convenient for use by designers and engineers. Inevitably, cost effective and accurate expressions are required to analyse the fin. In order to meet this demand, Adomian and Rach (1993) modified the Adomian decomposition method and introduced the double decomposition method (DDM).

Yang et al. (2008, 2010) solved the periodic base temperature in convective longitudinal fins using DDM, while Chiu and Chen (2003) applied the DDM to analyze convective-radiative fins.The method was found to have more advantages than the Adomian decomposition method, including faster convergence,reduced calculations, higher accuracy andprovision of a direct scheme for solving the non-linear problem without the need of linearization and iteration and most importantly, it gives an explicit form of solution to non-linear problem.It solves non-linear problems without linearization, perturbation, closure approximations, or discretization methods that could result in massive numerical computations. Therefore, in this present work, double decomposition method is applied to analyze thermal performance and optimum thermal design of convective straight fin with temperature-dependent thermal conductivity. The DMM is computationally convenient, provides analytic, direct scheme, verifiable solutions not requiring perturbation, linearization, or discretization and resulting massive computation. It gives faster convergence, reduced calculations, higher accuracy than ADM and more importantly, it gives an explicit form of solution to non-linear problem. Also, Golberg (1999) has shown that ADM does not converge in general, in particular, when the method is applied to linear operator equations. Furthermore, it was shown that Adomian's decomposition method is equivalent to Picard iteration method, and therefore it might diverge. From the previous studies and analysis, it was revealed that the DDM provides a very powerful, novel and accurate approximate analytical solution procedure that is applicable to a wide variety of linear and non-linear problems and thus makes it unnecessary to search for an auxiliary value that will satisfy second the boundary condition as in the case of HPM, HAM, ADM and VIM, and without searching for variational formulations in order to apply the finite element method for the problems and the difficulties associated with proper construction of the approximating functions for arbitrary domains or geometry of interest as in Galerkin weighted residual method (GWRM), least square method (LSM) and collocation method (CM) are overcome. Although, the method presents its own difficulty in determining the Adomian polynomials, A_m, the resulting solutions from the method are more physically realistic. It would be desirable to find easier ways of generating the Adomian polynomials and to study their properties to reduce the computational effort. From the present analysis, the results obtained by the method for solving the problem under investigation are compared with the exact solution for the linear problem and also with the numerical solution for the non-linear case and very good agreements were established.

2.0 PROBLEM FORMULATION

Consider a straight fin of temperature-dependent thermal conductivity *k(T)*, length L and thickness δ that is exposed on both faces to a convective environment at temperature T_∞ and with heat transfer co-efficient *h* shown in Figure1, assuming that the heat flow in the fin and its temperatures remain constant with time, the temperature of the medium surrounding the fin is uniform, the fin base temperature is uniform., there is no contact resistance where the base of the fin joins the prime surface, also the fin thickness is small compared with its width and length, so that temperature gradients acrossthe fin thickness and heat transfer from the edges of the fin may be neglected. The dimension *x* pertains to the length coordinatewhich has its origin at the tip of the finand has a positive orientation from the fin tip to the fin base. Following the model assumptions, the governing differential equation for the problem is shown in Equation (1).

$$\frac{d}{dx}\left[k(T)\frac{dT}{dx}\right]-\frac{h}{A_c}P(T-T_\infty)=0 \tag{1}$$

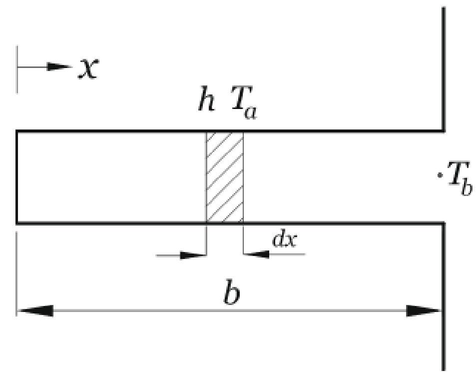


Figure 1. Schematic of the longitudinal straight fin geometry

The boundary conditions are

$$\begin{aligned} x &= b, \quad T = T_b \\ x &= 0, \quad \frac{dT}{dx} = 0 \end{aligned} \tag{2}$$

For many engineering applications, the thermal conductivity and the coefficient of heat transfer are temperature-dependent. Therefore, the temperature-dependent thermal properties and internal heat generation are given by

$$k\left(T\right) = k_a[1+\lambda(T-T_\infty)] \tag{3}$$

Substituting Equation (3) into Equation (1), we have

$$\frac{d}{dx}\left[k_a[1+\lambda(T-T_\infty)]\frac{dT}{dx}\right] - \frac{hP(T-T_\infty)}{A_c} = 0 \tag{4}$$

Introducing the following dimensionless parameters into Equation (4);

$$X = \frac{x}{b}, \quad \theta = \frac{T-T_\infty}{T_b-T_\infty}, \qquad K = \frac{k}{k_a}, \quad M^2 = \frac{PhL^2}{A_c k_a}, \quad \beta = \lambda(T_b - T_\infty) \tag{5}$$

One arrives at the dimensionless governing differential Equation (4) and the boundary conditions

$$\frac{d}{dX}\left[(1+\beta\theta)\frac{d\theta}{dX}\right] - M^2\theta = 0 \tag{6}$$

Equation (6) could be written in expanded form as

$$\frac{d^2\theta}{dX^2} + \beta\theta\frac{d^2\theta}{dX^2} + \beta\left(\frac{d\theta}{dX}\right)^2 - M^2\theta = 0 \tag{7}$$

where the boundary conditions are

$$\begin{aligned} X &= 1, \quad \theta = 1 \\ X &= 0, \quad \frac{d\theta}{dX} = 0 \end{aligned} \tag{8}$$

3.0 METHOD OF SOLUTION: DOUBLE DECOMPOSITION METHOD

The nonlinearity in the governing Equation (7) makes it very difficult to generate a closed form solution for Equation (7). Therefore, recourse has to be made to either approximation analytical methods, semi-numerical methods or numerical methods of solution. In this work, an approximate analytical method of solution, double decomposition method is used. It makes the calculation accuracy much higher than the Adomian decomposition and lowers the computational load. The double decomposition method uses the same operator as the Adomian decomposition method, but decomposes the first undefined parameters. To do this, the zero-order decomposition formula is set into the boundary conditions and then evaluates the undefined parameters. The procedure of the method is described as follows:

The general nonlinear equation is in the form

$$Lu + Nu + Ru = g \tag{9}$$

The linear terms are decomposed into L + R, with L taken as the highest order derivative which is easily invertible and R as the remainder of the linear operator of less order than L. where g is the system input or the source term and u is the system output, Nu represents the nonlinear terms, which is assumed to be analytic. L^{-1} is regarded as the inverse operator of L and is defined by a definite integration from *0* to *x*, i.e.

$$[\mathrm{L}^{-1} f](x) = \int_0^x f(v)dv \tag{10}$$

If L is a second-order operator, then L^{-1} is a two ford indefinite integral i.e. L^{-1} could be expressed as

$$[\mathrm{L}^{-1} f](x) = \int_1^x \int_0^x f(v)dvdv \tag{11}$$

Applying the inverse operator L^{-1} to the both sides of Equation (9), and using the given conditions, the resulting equation could be written as

$$u = \mu(x) - L^{-1}Ru - L^{-1}Nu \tag{12}$$

Where $\mu(x) = \lambda_x + L^{-1}g$ and λ_x represents the term arising from integrating the source term g(x).

The Adomian methods decomposes the solution *u(x)* into a series

$$u = \sum_{m=0}^{\infty} u_m \tag{13}$$

and the nonlinear term into a series

$$Nu = \sum_{m=0}^{\infty} A_m \tag{14}$$

where A_m*'s* are Adomian's polynomials of u_0 ,u_1, . . ., u_m and are obtained for the nonlinearity
*Nu = f(u)*from the recursive formula

$$A_m = \frac{1}{m!}\left[\frac{d^m}{d\zeta^m}[\, fu(\zeta)]\right]_{\zeta=0} = \frac{1}{m!}\left[\frac{d^m}{d\zeta^m} f\left(\sum_{i=0}^{\infty}\zeta^i y_i\right)\right]_{\zeta=0} \qquad m = 0,1,2,3,\ldots \tag{15}$$

where ζ is a grouping parameter of convenience.

Using the double decomposition, the integral term $\lambda_x$could be further decompose as

$$\lambda_x = \sum_{m=0}^{\infty} \lambda_{x,m} \tag{16}$$

Substituting Eqs. (13), (14) and (16) into Equation (12), we have

$$\sum_{m=0}^{\infty} u_m = \sum_{m=0}^{\infty} \lambda_{x,\mathrm{m}} + L^{-1}g - L^{-1}R\sum_{m=0}^{\infty} u_m - L^{-1}\sum_{m=0}^{\infty} A_m \tag{17}$$

Assuming that $\lambda_{x,m} = a_{o,m} + xa_{1,m}$ .The constants of integration $a_{0,m}$ *and* $a_{1,m}$ can be found from the boundary conditions

Therefore, from the established recursive relation in Equation (17), one can write the double decompositions solution as

$$u_0 = a_{0,0} + xa_{1,0} + L^{-1}g$$

$$u_1 = a_{0,1} + xa_{1,1} - L^{-1}Ru_0 - L^{-1}A_0$$

$$u_2 = a_{0,2} + xa_{1,2} - L^{-1}Ru_1 - L^{-1}A_1$$

$$u_3 = a_{0,3} + xa_{1,3} - L^{-1}Ru_2 - L^{-1}A_2$$

$$u_4 = a_{0,4} + xa_{1,4} - L^{-1}Ru_3 - L^{-1}A_3$$

$$\vdots$$

$$u_n = a_{0,\mathrm{n}} + xa_{1,\mathrm{n}} - L^{-1}Ru_{n-1} - L^{-1}A_{n-1}$$

(18)

While the solution obtained by decomposition is generally an infinite series, an *(n+1)* terms approximation φ_m to θ usually serves as the practical solution. This could be written as

$$\varphi_{n+1} = \sum_{m=0}^{\infty} u_m = u_0 + u_1 + u_2 + u_3 + \ldots + u_n \qquad n \geq 1 \tag{19}$$

Such that $\lim_{n\to\infty} \varphi_{n+1} = \theta$

3.1 The Fin temperature distribution

From the Adomian decomposition analysis, the linear operator is defined as

$$L_X = \frac{d}{dX} \tag{20}$$

Substituting Equation (20) into Equation (7), we have

$$L_X\theta = M^2\theta - \beta\theta\frac{d^2\theta}{dX^2} - \beta\left(\frac{d\theta}{dX}\right)^2 \tag{21}$$

Equation (21) could also be written as

$$L_X\theta = M^2\theta - \beta NA - \beta NB \tag{22}$$

where the nonlinear terms

$$NA = \theta \frac{d^2\theta}{dX^2} = \sum_{m=0}^{\infty} A_m \qquad (23a)$$

$$NB = \left(\frac{d\theta}{dX}\right)^2 = \sum_{m=0}^{\infty} B_m \qquad (23b)$$

Using Equation (15) the *Ai's* and *Bi's* are expressed as

$$A_0 = \theta_0 \frac{d^2\theta_0}{dX^2}$$

$$A_1 = \theta_1 \frac{d^2\theta_0}{dX^2} + \theta_0 \frac{d^2\theta_1}{dX^2}$$

$$A_2 = \theta_2 \frac{d^2\theta_0}{dX^2} + \theta_1 \frac{d^2\theta_1}{dX^2} + \theta_0 \frac{d^2\theta_2}{dX^2}$$

$$A_3 = \theta_3 \frac{d^2\theta_0}{dX^2} + \theta_2 \frac{d^2\theta_1}{dX^2} + \theta_1 \frac{d^2\theta_2}{dX^2} + \theta_0 \frac{d^2\theta_3}{dX^2}$$

$$A_4 = \theta_4 \frac{d^2\theta_0}{dX^2} + \theta_3 \frac{d^2\theta_1}{dX^2} + \theta_2 \frac{d^2\theta_2}{dX^2} + \theta_1 \frac{d^2\theta_3}{dX^2} + \theta_0 \frac{d^2\theta_4}{dX^2}$$

$$\vdots$$

$$A_m = \theta_m \frac{d^2\theta_0}{dX^2} + \theta_{m-1} \frac{d^2\theta_1}{dX^2} + \theta_{m-2} \frac{d^2\theta_2}{dX^2} + \theta_{m-3} \frac{d^2\theta_3}{dX^2} + \ldots + \theta_0 \frac{d^2\theta_m}{dX^2} \qquad (24)$$

and

$$B_0 = \left(\frac{d\theta_0}{dX}\right)^2$$

$$B_1 = 2 \frac{d\theta_0}{dX} \frac{d\theta_1}{dX}$$

$$B_2 = \left(\frac{d\theta_1}{dX}\right)^2 + 2\frac{d\theta_0}{dX}\frac{d\theta_2}{dX}$$

$$B_3 = 2\frac{d\theta_1}{dX}\frac{d\theta_2}{dX} + 2\frac{d\theta_0}{dX}\frac{d\theta_3}{dX}$$

$$B_4 = \left(\frac{d\theta_2}{dX}\right)^2 + 2\frac{d\theta_1}{dX}\frac{d\theta_3}{dX} + 2\frac{d\theta_0}{dX}\frac{d\theta_4}{dX}$$

$$\vdots$$

(25)

If we operate L_X^{-1} on both sides of Equation (22), we obtained

$$L_X^{-1}L_X\theta = M^2 L_X^{-1}\theta - \beta L_X^{-1}NA - \beta L_X^{-1}NB \tag{26}$$

which gives

$$\theta = \theta_0 + M^2 L_X^{-1}\theta - \beta L_X^{-1}NA - \beta L_X^{-1}NB \tag{27}$$

where the inverse operator $\mathrm{L}^{-1}(\bullet) = \int_1^X \int_0^X (\bullet)dXdX$

The value of the first term can be determined as

$$\theta_0 = a_{0,0} + xa_{1,0} \tag{28}$$

where the constants $a_{0,0}$ *and* $a_{1,0}$ can be found from the boundary conditions in Equation (8).

$$\theta_1 = a_{0,1} + xa_{1,1} + M^2 L_X^{-1}\theta_0 - \beta L_X^{-1}NA_0 - \beta L_X^{-1}NB_0$$

$$\theta_2 = a_{0,2} + xa_{1,2} + M^2 L_X^{-1}\theta_1 - \beta L_X^{-1}NA_1 - \beta L_X^{-1}NB_1$$

$$\theta_3 = a_{0,3} + xa_{1,3} + M^2 L_X^{-1}\theta_2 - \beta L_X^{-1}NA_2 - \beta L_X^{-1}NB_2 \tag{29}$$

Generally, the recursive relationship in Equation (30) can be used to determine the iterates

$$\theta_{m+1} = a_{0,\mathrm{m+1}} + xa_{1,\mathrm{m+1}} + M^2 L_X^{-1}\theta_m - \beta L_X^{-1}NA_m - \beta L_X^{-1}NB_m \tag{30}$$

From Eqs. (28) and (29), the first four iterations are given as

$$\theta_0 = 1$$

$$\theta_1 = \frac{M^2X^2}{2} - \frac{M^2}{2}$$

$$\theta_2 = \frac{5M^4}{24} + \frac{M^2\beta}{2} + \frac{M^4X^4}{24} - \frac{M^4X^2}{4} - \frac{M^2X^2\beta}{2}$$

$$\theta_3 = \frac{M^6X^6}{720} - \frac{M^6X^4}{48} + \frac{7M^6}{360} - \frac{5M^6X^4}{24} - \frac{13M^4\beta}{24} - \frac{3M^4\beta X^2}{4} + \frac{5M^4X^2}{48} - \frac{5M^4}{48} - \frac{\beta M^2X^2}{2} - \frac{\beta M^2}{2} + \frac{\beta^2M^2X^2}{2} - \frac{\beta^2M^2}{2}$$

$$\vdots$$

(31)

Summing up the iterates, gives

$$\varphi_m = \sum_{m=0}^{m-1}\theta_m = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_{m-1} \tag{32}$$

Therefore, the components of θ are determined and are written as m-terms approximation such that $\operatorname*{Lim}_{m\to\infty}\varphi_m = \theta$

4.0 FIN PARAMETERS FOR THERMAL PERFORMANCE INDICATIONS

The performance indication parameters for fin includes heat transfer rate at the base of the fin, the total heat flux from the fin, the efficiency and the effectiveness of the fin. In this section, each thermal performance indication parameter is analyzed as follows.

4.1 Heat flux of the Fin

The fin base heat flux is given by

$$q_{bn} = A_c k(T) \frac{dT}{dx} \tag{33}$$

The dimensionless heat transfer rate at the base of the fin (X=1) is obtained as

$$q_b = \frac{q_{bn} L}{k_a A_c (T_b - T_\infty)} = (1 + \beta\theta) \frac{d\theta}{dX} \tag{34}$$

The total heat flux of the fin is given by

$$q_T = \frac{q_b}{A_c h(T - T_b)} \tag{35}$$

Substituting Equation (34) and introducing the dimensionless parameters in Equation (5) into Equation (35), gives

$$q_T = \frac{1}{Bi} \frac{k(\theta)}{h} \frac{d\theta}{dX} = \frac{1}{Bi} (1 + \beta\theta) \frac{d\theta}{dX} \tag{36}$$

4.2 Fin efficiency

The amount of heat dissipated from the entire fin is found by using Newton's law of cooling as

$$q_f = \int_0^1 Ph(T - T_\infty) dX \tag{37}$$

The maximum heat dissipated is obtained if the fin base temperature is kept throughout the fin

$$q_{\max} = PhL(T_b - T_\infty) \tag{38}$$

Fin efficiency is defined as the ratio of the fin heat transfer rate to the rate that would be if the entire fin were at the base temperature and is given by

$$\eta = \frac{q_f}{q_{\max}} = \frac{\int_0^L Ph(T - T_\infty) dx}{PhL(T_b - T_\infty)} \tag{39}$$

Therefore, the fin efficiency in dimensionless variables is given by

$$\eta = \int_0^1 \theta(\mathrm{X})dX \tag{40}$$

It is very important to point out that the thermo-geometric parameter or the fin performance factor, M could be written in terms of Biot number, Bi and the aspect ratio, a_r as shown in Equation (41).

$$M^2 = \frac{Ph_bL^2}{A_ck_a} = \frac{(2L)h_bL^2}{(L\delta)k_a} = \frac{2h_b\delta L^2}{\delta^2 k_a} = \frac{2h_b\delta}{k_a}\left(\frac{L}{\delta}\right)^2 = 2Bia_r^2 \tag{41}$$

where

$$Bi = \frac{h_b\delta}{k_a},\ a_r = \frac{L}{\delta}$$

From Equation (41), it implies that

$$M = a_r\sqrt{2Bi} \tag{42}$$

4.3 Fin effectiveness

The removal number or fin effectiveness is the ratio of the fin dissipation (equal, in the steady state, to the heat passing through the base of the fin by conduction) to the heat passing through the fin footprint of the base or prime surface if the fin were not present. Following the definition, the effectiveness of the fin could be expressed mathematically as

$$\varepsilon = \frac{q_f}{q_{fb}} \tag{43}$$

where q_{fb} is the amount of heat dissipation from the area of the fin base and is given by

$$q_{fb} = Ph_b\frac{\delta}{2}(T_b - T_\infty) \tag{44}$$

Substituting Equation (37) and (44) into Equation (43), gives

$$\varepsilon = \frac{q_f}{q_{fb}} = \frac{\int_0^L 2Ph(T - T_\infty)}{Ph\delta(T_b - T_\infty)} \tag{45}$$

Therefore, the fin effectiveness in dimensionless variables is given by

$$\varepsilon = 2a_r \int_0^1 \theta(\mathrm{X})\, dX \tag{46}$$

5.0 FIN OPTIMIZATION

The optimization of the fin could be achieved either by minimizing the volume (weight) for any required heat dissipation or maximizing the heat dissipation for any given fin volume. The later approach is adopted in this work.
The constant fin volume is defined as $V=A_c b$. Following Equation (37), one can therefore write the heat dissipation per unit volume as

$$\frac{q_f}{V} = \frac{\int_0^L Ph(T - T_\infty)dx}{A_c b} \tag{47}$$

The dimensionless form of Equation (47) is given as

$$Q_f = \frac{q_f}{k_a(\mathrm{T}_b - \mathrm{T}_a)}\left(\frac{A_p}{V}\right) = \frac{PhA_p \int_0^1 \theta dX}{k_a A_c} \tag{48}$$

Equation (48) could be written as

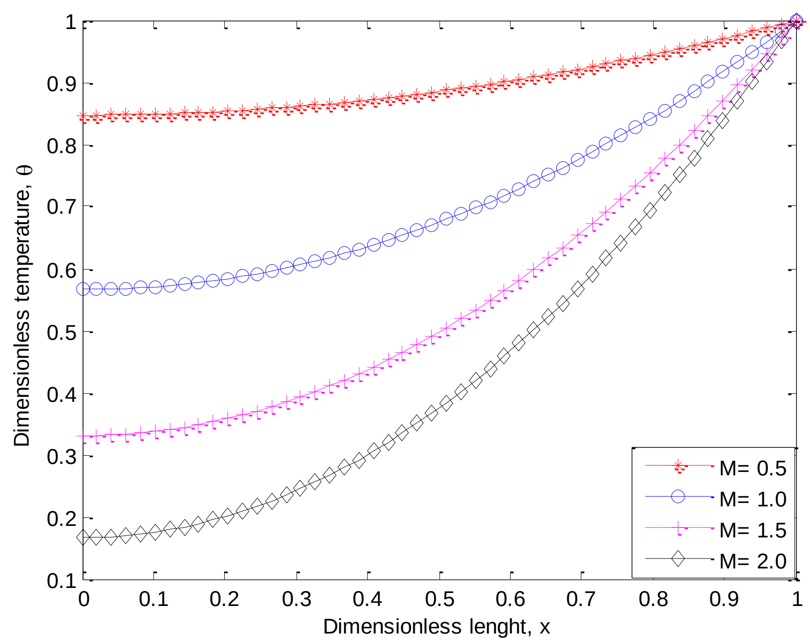
$$Q_f = \xi M^{2/3} \int_0^1 \theta dX \tag{49}$$

where $A_p = \delta b \qquad \xi = \left(\dfrac{2h\sqrt{A_p}}{k_a}\right)^{2/3}$

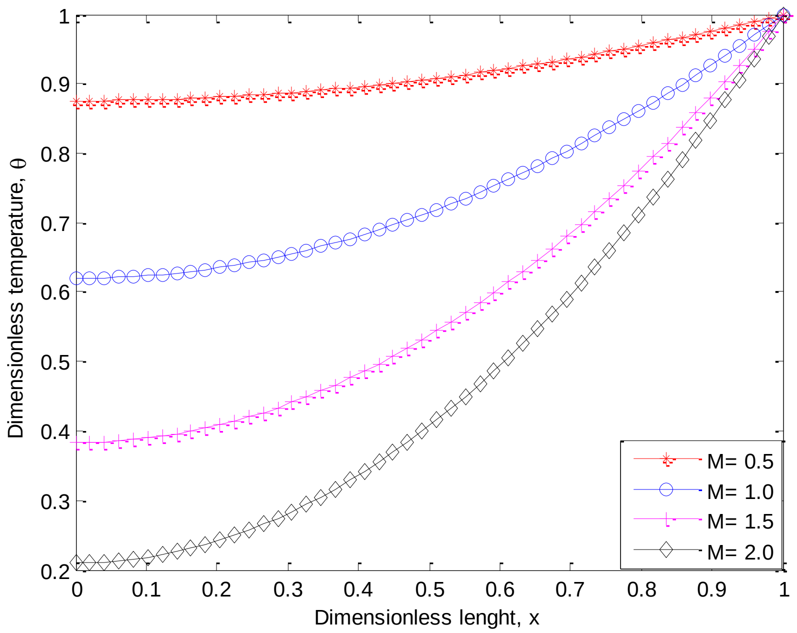
The maximum heat dissipation value occurs at the condition when the optimum fin characteristics have been achieved. The fin dimensions in this situation represent the optimum fin configuration per unit volume. With the volume constant, the optimization procedure is also carried out to fix the profile area A_p by first expressing $\frac{Q_f}{\xi}$ as a function of the thermo-geometric parameter, M (or fin length, b) and then searching for the optimum value of M or b.

6.0 RESULTS AND DISCUSSIONS

Figures 2a and 2b show the variation of dimensionless temperature with dimensionless length of the fin and also depict the effect of the thermogeometric parameter on the straight fin with an insulated tip. It is shown that as the thermogeometric parameter increases, the rate of heat transfer (the convective heat transfer) through the fin increases as the temperature in the fin drops faster (becomes steeper reflecting high base heat flow rates) as depicted in the figures. It can be inferred from the results that the ratio of convective heat transfer to conductive heat transfer has much effect on the temperature distribution, rate of heat transfer at the base of the fin, efficiency and effectiveness of the fin. As *h* increases (or k_b decreases), the ratio h/k_b increases at the base of the fin and consequently the temperature along the fin, especially at the tip of the fin decreases i.e. the tip end temperature decrease as M increases. The profile has steepest temperature gradient at *M=1.0*, but it is much higher value gotten from the lower value of thermal conductivity than the other values of *M* in the profiles produces a lower heat-transfer rate.

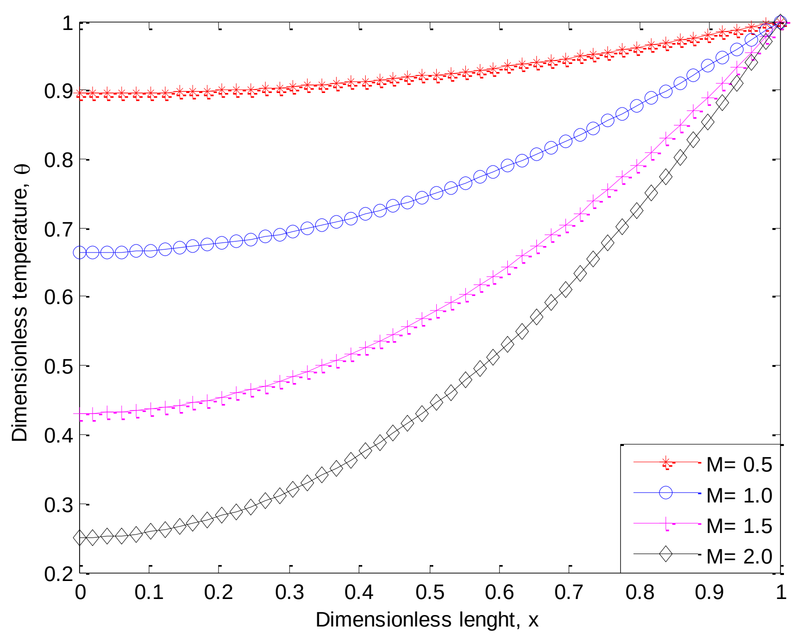


(a)



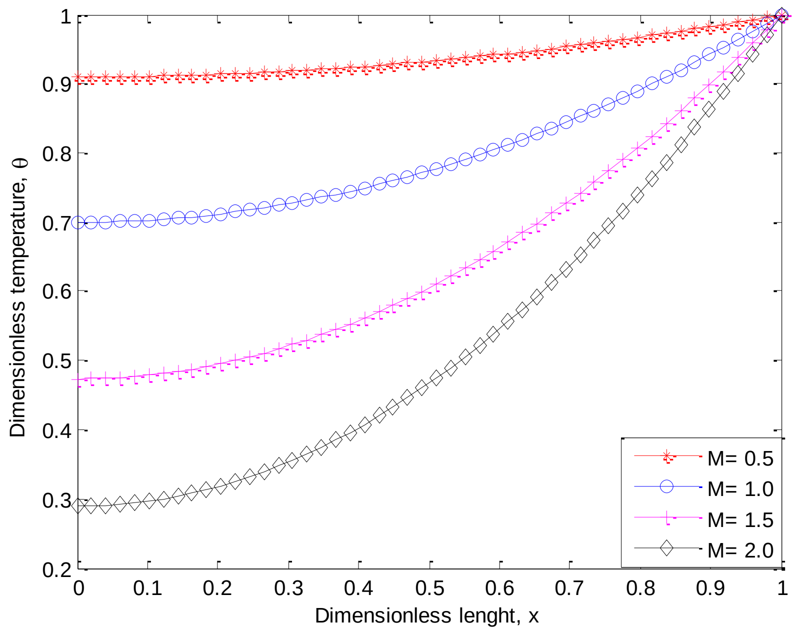
Dimensionless temperature, θ
Dimensionless lenght, x
M= 0.5
M= 1.0
M= 1.5
M= 2.0

(b)



Dimensionless temperature, θ
Dimensionless lenght, x
M= 0.5
M= 1.0
M= 1.5
M= 2.0

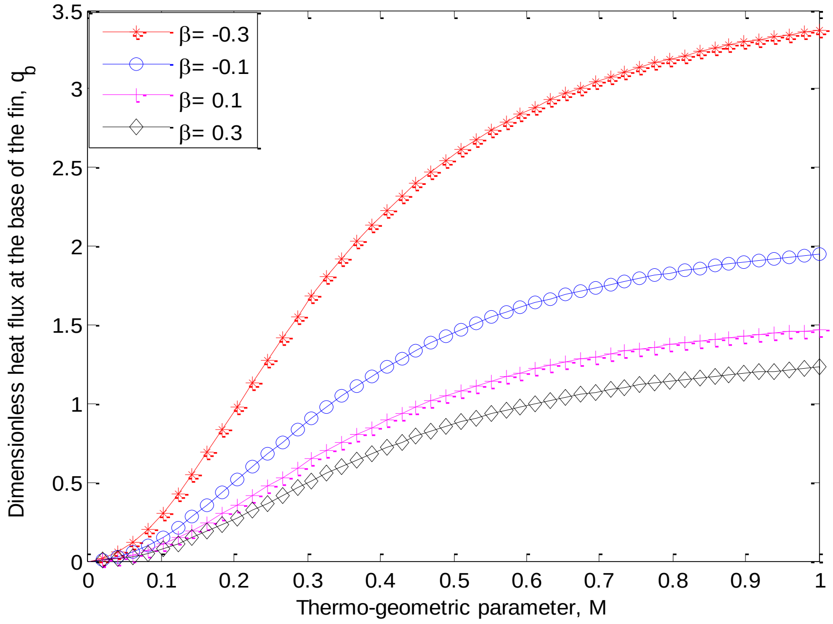
(c)



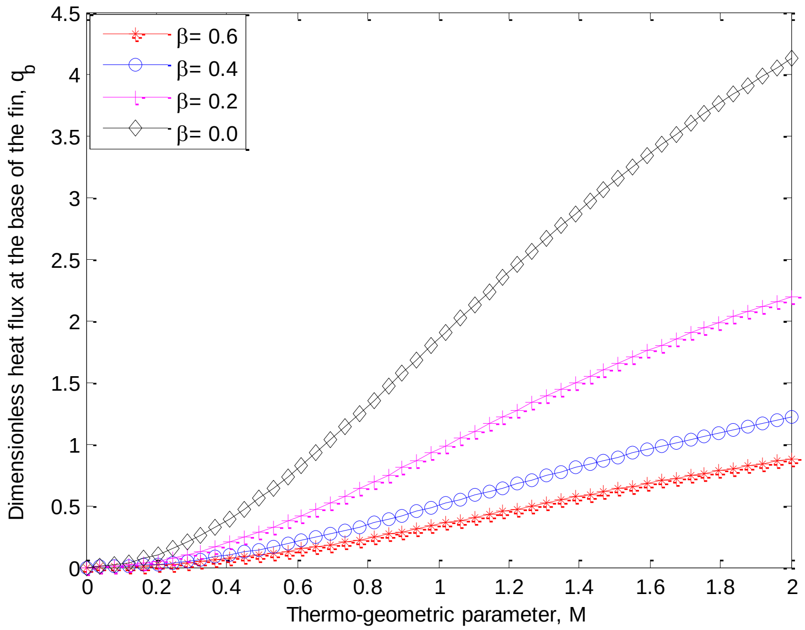
(d)

Figure 2. Effects of thermo-geometric parameter on the temperature distribution in the fin when (a) β= -0.3 (b) β=-0.1(c) β=0.1 (d) β=0.3

This shows that the thermal performance or efficiency of the fin is favoured at low values of thermogeometric parameter since the aim (high effective use of the fin) is to minimize the temperature decrease along the fin length, where the best possible scenario is when $T=T_b$ everywhere. One of the major important analyses in the fin problem is the determination of the rate of heat transfer at the base of the fin. **Figure 3a** shows the effects of no-linear or thermal conductivity term on the dimensionless heat transfer rate at the base of the fin while **Figure 4** shows the effects of no-linear or thermal conductivity term on the dimensionless total heat flux of the fin. Also, the figures depict the variation of the rate of heat transfer with the thermo-geometric parameter. It could be deducted that the thermal conductivity and the thermo-geometric parameter have direct and significant effects on the rate of heat transfer at the base of the fin. Thus, the operational parameters must be carefully chosen to ensure that the fin retains its primary purpose of removing heat from the primary surface.

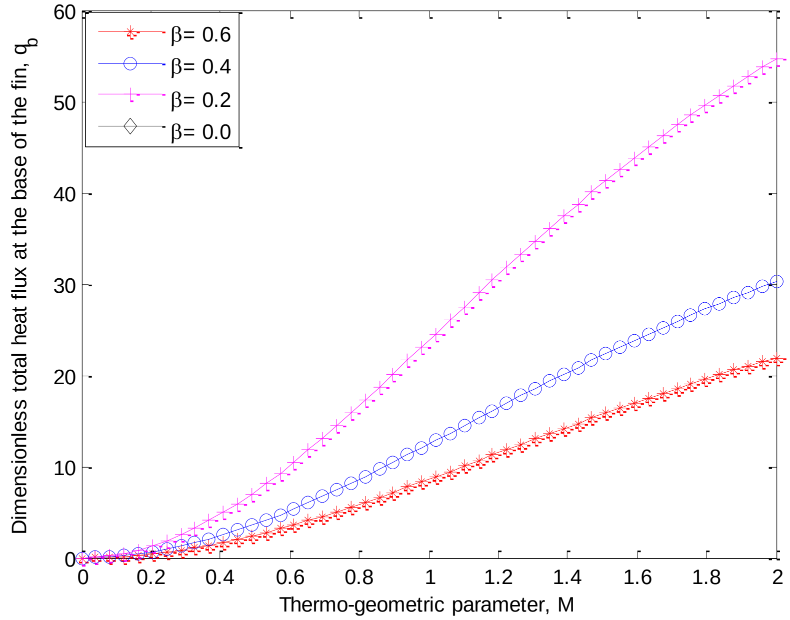


(a)

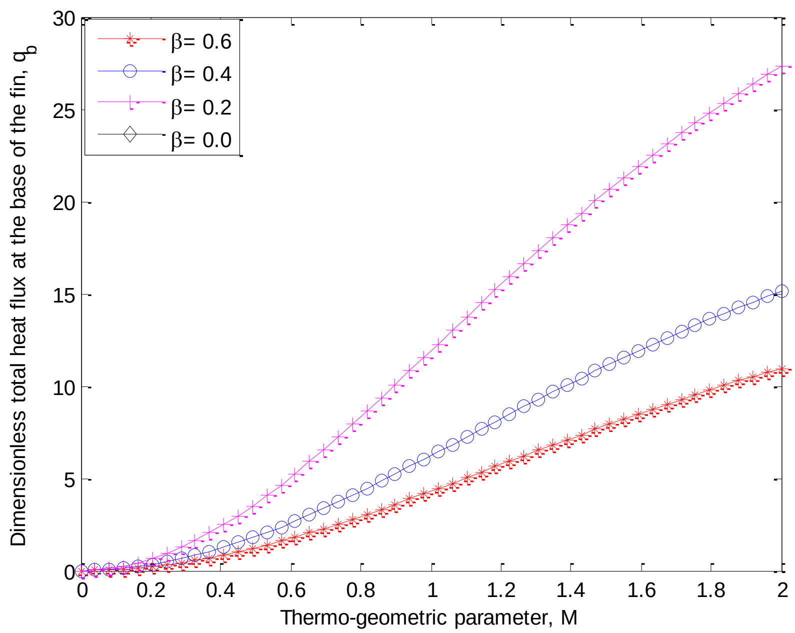


(b)

Figure 3. Effects of thermal conductivity parameter on heat transfer rate at the base of the fin



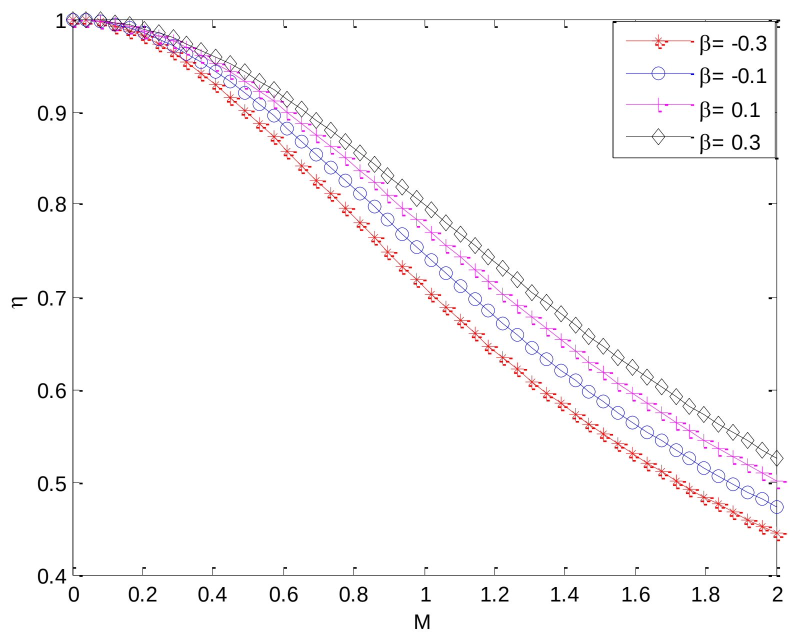
(a)



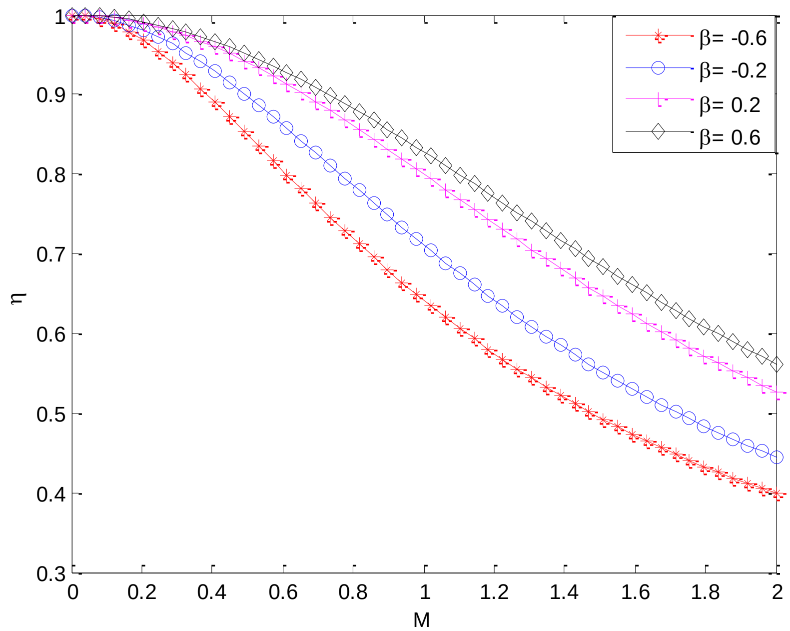
(b)

Figure 4. Effects of non-linear thermal conductivity parameter on the dimensionless total heat flux of the fin (a), Bi=0.04 (b) Bi=0.08

Figure 5a and 5b show the effects of non-linear thermal conductivity and thermo-geometric parameters on the fin efficiency. The figures show that the fin efficiency decreases monotonically with increasing thermogeometric parameter. Also, it shows the variation of fin efficiency with thermogeometric in longitudinal convecting fin with insulated tip. From the figures, it is shown that as the thermogeometric parameter increases, the efficiency of the fin decreases. When the thermogeometric fin parameter equals to zero, the fin efficiency is 100%, which implies that there is no conduction resistance or no presence of fin at all. As the convective heat transfer coefficient to thermal conductivity ratio approaches zero, the temperature at every point in the fin is equal to the temperature of the base. The inverse variation in the fin efficiency with the thermo-geometric parameter is due to the fact that as more material is attached to the prime surface, the resistance to heat flow increases thereby reducing the fin efficiency. Upon further increase in the fin thermo-geometric parameter, the effect of reducing the resistance becomes visible in the sense that the fin efficiency starts to normalize. Therefore, high efficiency of the fin could be achieved by using small values of thermogeometric parameter, which could be realized using a fin of small length or by using a material of better thermal conductivity. Moreover, the results depicted that care must be taken in the choice of length of fin used during applications. This is because, thermogeometric parameter (which increases as the fin length increases) tends to infinity, and the fin efficiency tends to zero. The fin to a large extent of its length will remain at ambient. This consequently results in weak conduction limit. The extended area is largely useless in the heat transfer process and hence inefficient. Therefore, very long fins are to be avoided in practice.



(a)

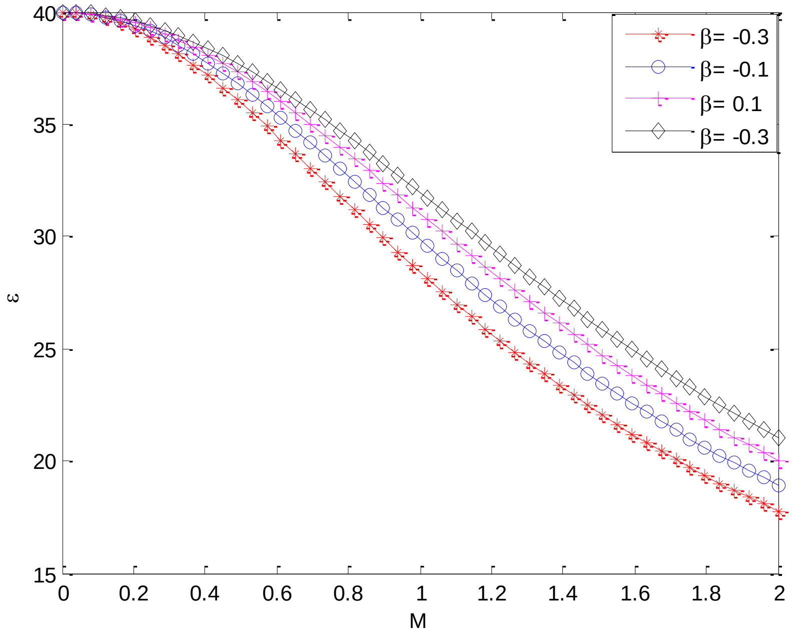


(b)

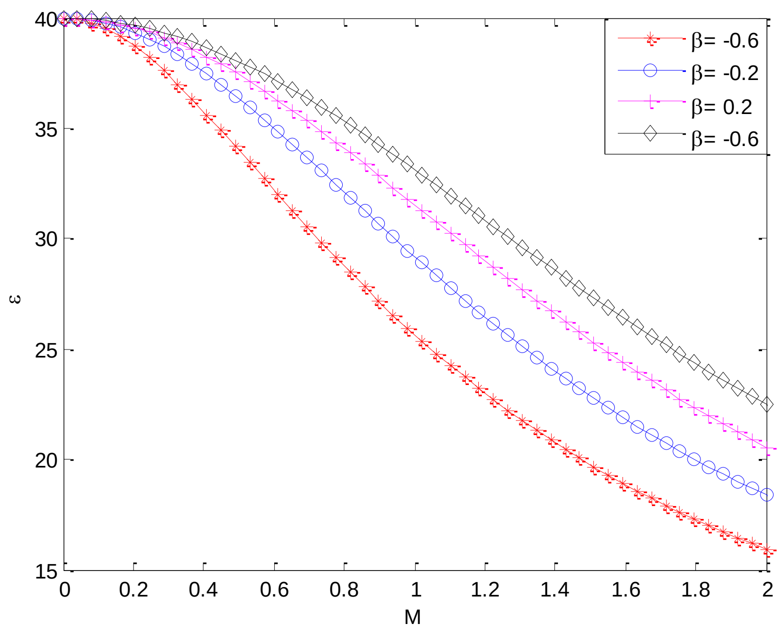
Figure 5. Effects of non-linear thermal conductivity and thermo-geometric parameters on the efficiency of the fin

Also, as shown in the Figures 5a and 5b, the fin efficiency is unity in the limit M→0. In this limit, the actual heat transfer rate from the fin is zero. This fin parameter (the thermo-geometric parameter) plays a very important role in determining the amount of heat transfer from the fin as it accounts for the effects of decrease in temperature on the heat transfer from the fin. Since, the fin temperature drops along the fin length, the fin efficiency decreases with increase in fin length. Therefore, in practice required fin length should be properly determined because the fin length that causes the fin efficiency to drop below 60% usually cannot be justified economically and should be avoided.

Figures 6a and 6b show the effects of non-linear thermal conductivity and thermos-geometric parameters (under the aspect ratio of 20) on the effectiveness of the fin. As the aspect ratio increases, higher local temperature is produced in the fin, thereby increases the effectiveness of the fin. Also, it is shown that high effectiveness of fin could be achieved by using small values of thermogeometric parameter and this could be realized using a fin of small length or by using a material of better thermal conductivity.

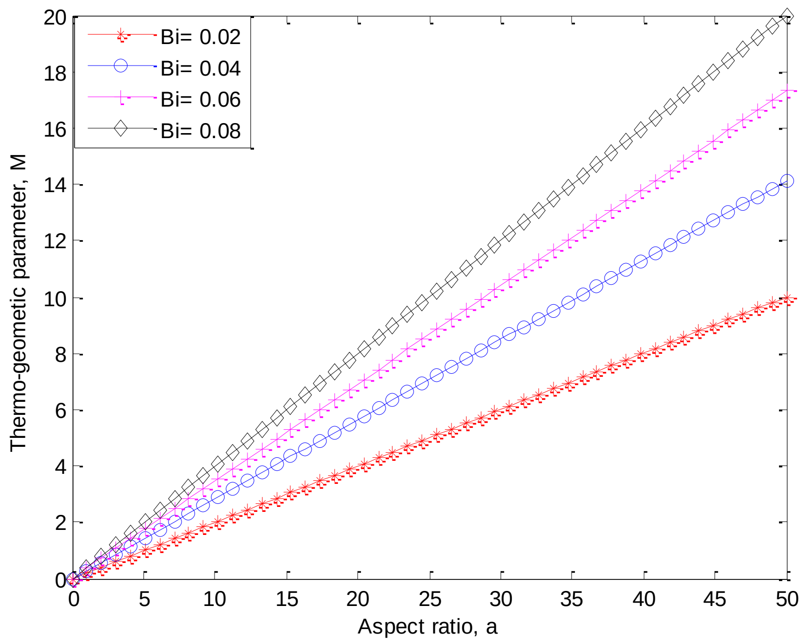


(a)

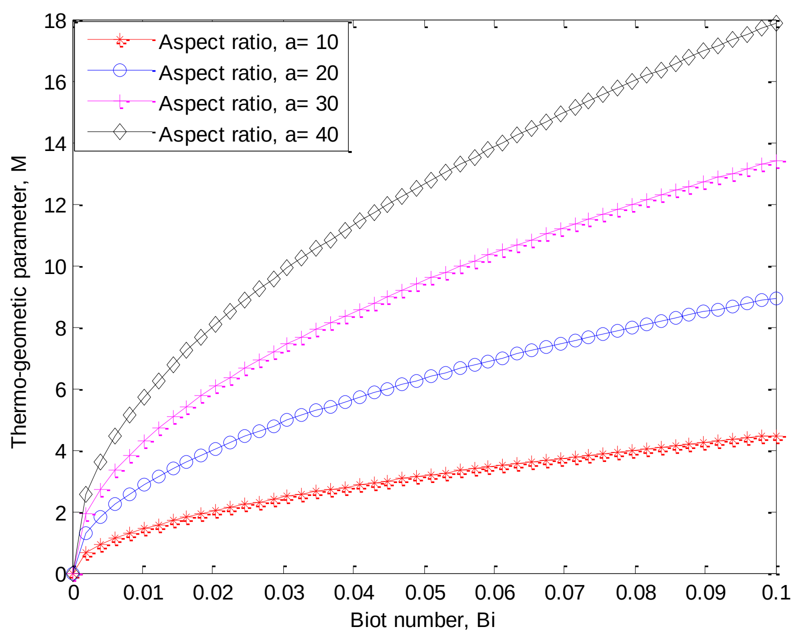


(b)

Figure 6. Effects of non-linear thermal conductivity and thermo-geometric parameters on the effectiveness of the fin



(a)



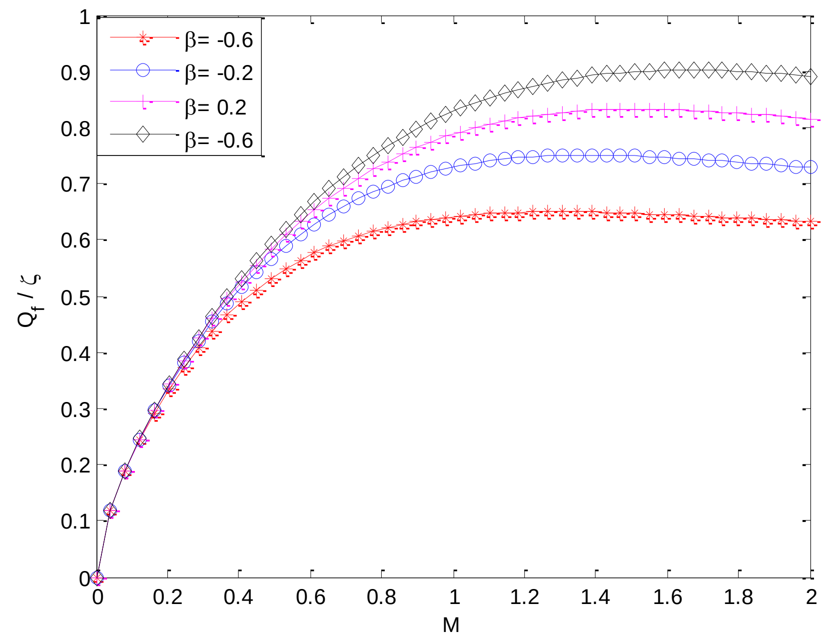
(b)

Figure 7. Effects of Biot number on the thermo-geometric parameter of the fin

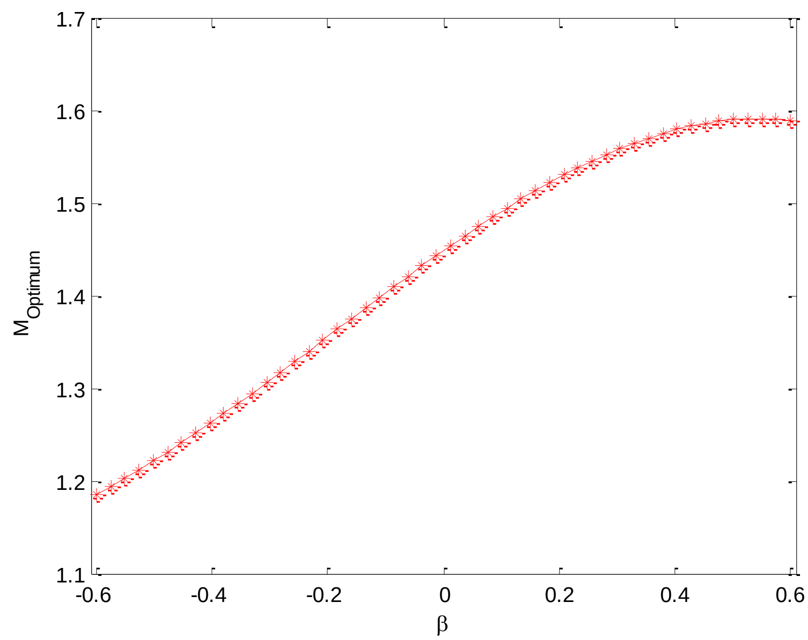
The effects of Biot number and aspect ratio on the thermo-geometric parameter (the fin performance factor) are shown in Figure 7. The fin performance factor increases as the aspect ratio and Biot number increase. However, the thermal performance or efficiency of the fin is favoured at low values of thermogeometric parameter since the aim (high effective use of the fin) is to minimize the temperature decrease along the fin length, where the best possible scenario is when $T=T_b$ everywhere. It must be pointed out that Equation (42) shows the direct relationship between the thermogeometric parameter, M and the Biot number, Bi which directly depends on the fin length. A small value of M corresponds to a relatively short and thick fin of poor thermal conductivity and a high value of M implies a long fin or fin with low value of thermal conductivity. Since, the thermal performance or efficiency of the fin is favoured at low values of thermogeometric fin parameter, very long fins are to be avoided in practice. A compromise is reached for one-dimensional analysis of fins $0 < Bi < 0.1$. When the Biot number is greater than 0.1, two dimensional analysis of the fin is recommended as one-dimensional analysis predicts unreliable results for such limit.

Figure 8a shows the nondimensional heat transfer Q/ζ (for a unit fin volume) varying with M from 1 and 2 for specified values of non-linear thermal conductivity terms, β, under a given profile area, A_p, the heat transfer first rises and then falls as the fin length increases. Also, as the optimum fin length (at which Q/ζ reaches a maximum value) increases as the non-linear thermal conductivity term, β increases. It also shows that the optimum value of M can be obtained based upon the value of non-linear term. Therefore, from the analysis the optimum dimensions of the convection fin with variable thermal conductivity is established and the relative values of optimum M and β are shown in Figure 8b.

The approximate analytical method of solution was validated by the exact solution in Figure 9a and 9b for the linear thermal model of the fin problem and the non-linear problem was validated with numerical solution as shown in Figures 10a and 10b. It is depicted that the double decomposition method is highly accurate and agrees very well with the exact solution for the linear problem and also with the numerical solution for non-linear problem.



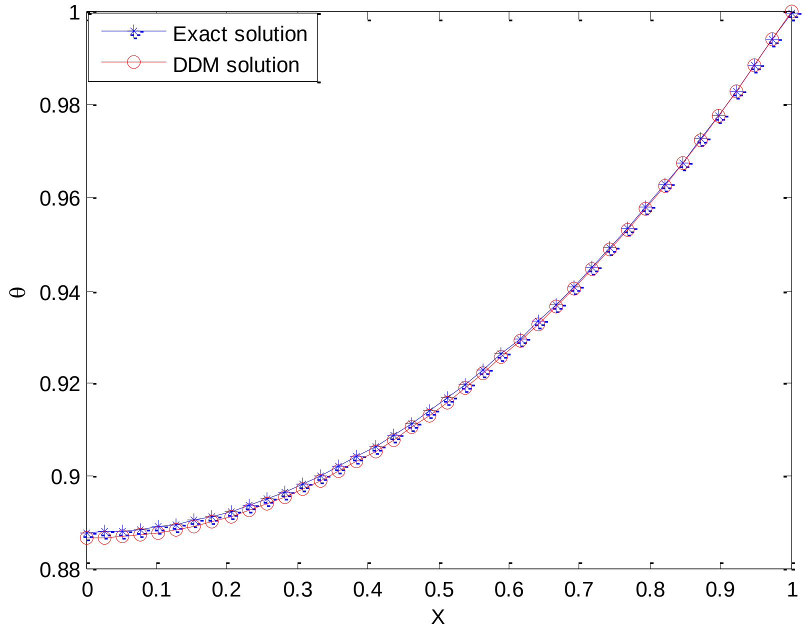
(a)



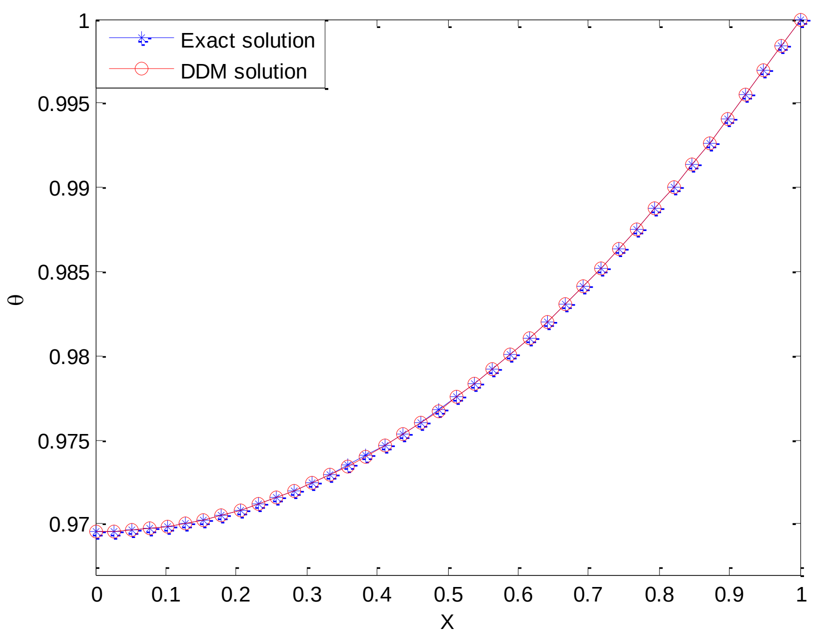
(b)

Figure 8 (a) Effects of non-linear thermal conductivity and thermo-geometric parameters on the dimensionless heat transfer, Q_f/ζ (b) Effects of non-linear thermal conductivity parameter on the optimum thermo-geometric parameter

6.1 Validation of results and thermal stability analysis of the fin

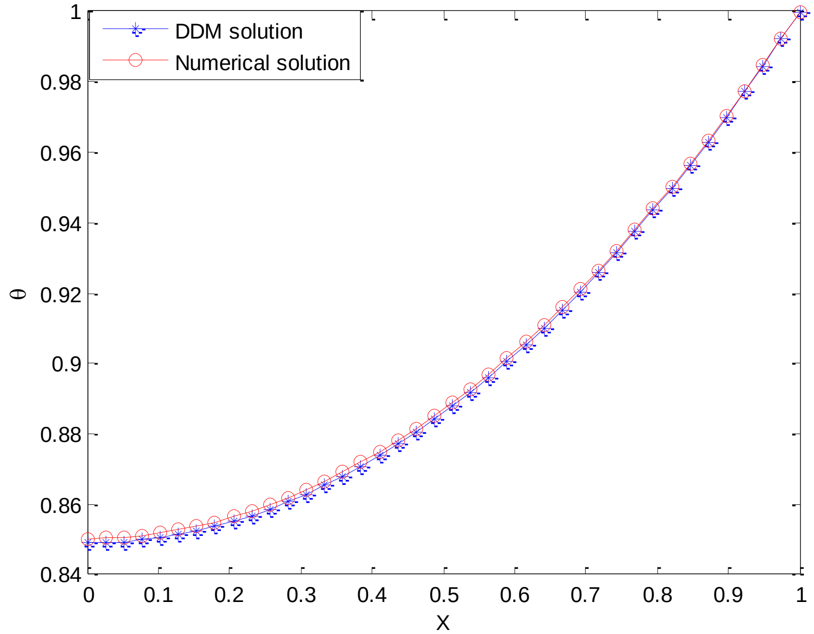


(a)

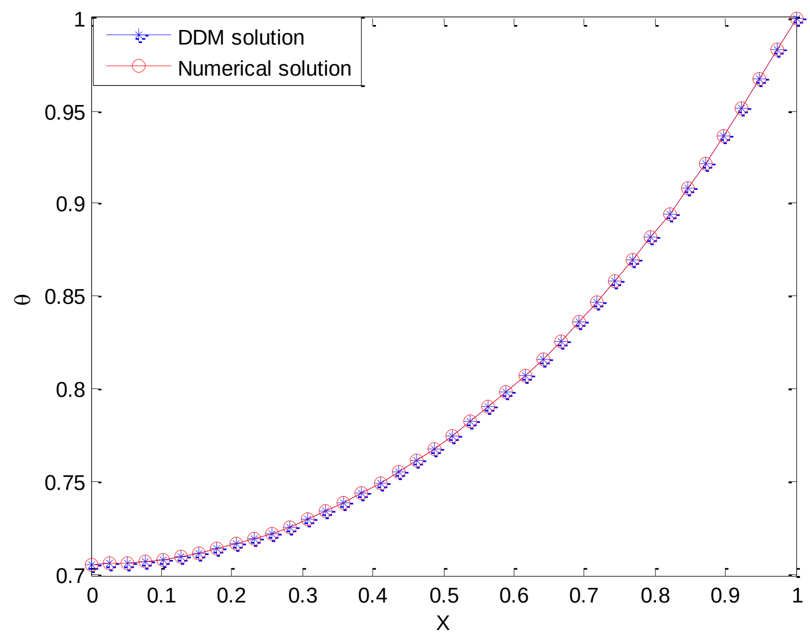


(b)

Figure 9. Validation of the results when (a) M=0.25, β=0 (a) M=0.5, β=0



(a)



(b)

Figure 10. Validation of the results when (a) M=0.5, β=-0.3 (b) M=1.0, β= 0.4

7.0 CONCLUSIONS

In this paper, thermal performance study and optimum design analysis of straight fin with variable thermal conductivity have been carried out using double decomposition method. The analysis revealed that the fin temperature distribution, the total heat transfer, the fin effectiveness, and the fin efficiency are significantly affected by the thermo-geometric and thermal parameters of the fin. Also, it is established that the optimum fin length increases as the non-linear thermal conductivity term, increases. It also shows that the optimum value of M can be obtained based upon the value of non-linear term.Therefore, the operational parameters must be carefully chosen to ensure that the fin retains its primary purpose of removing heat from the primary surface. The results obtained in this analysis provides platform for improvement in the design of fin in heat transfer equipment.

Nomenclature

a_r aspect ratio
A_c cross sectional area of the fins
A_p profile area of the fins
B Lenght of the fin
Bi Biot number
h heat transfer coefficient
k thermal conductivity of the fin material
k_a thermal conductivity of the fin material at ambient temperature
k_b thermal conductivity of the fin material at the base temperature of the fin
K dimensionless thermal conductivity of the fin material
M dimensionless thermo-geometric fin parameter
m^2 thermo-geometric fin parameter
P perimeter of the fin
T Temperature
T_∞ ambient temperature
T_b Temperature at the base of the fin
x fin axial distance, m
X dimensionless length of the fin
q rate of heat transfer
Q_f dimensionless heat transfer

Greek Symbols

β thermal conductivity parameter or non-linear parameter
δ thickness of the fin, m
θ dimensionless temperature
θ_b dimensionless temperature at the base of the fin
η efficiency of the fin
ε effectiveness of the fin

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